

GRAHAM-WITTEN'S CONFORMAL INVARIANT FOR CLOSED FOUR DIMENSIONAL SUBMANIFOLDS

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ABSTRACT. It was proved by Graham and Witten in 1999 that conformal invariants of submanifolds can be obtained via volume renormalization of minimal surfaces in conformally compact Einstein manifolds. The conformal invariant of a submanifold Σ is contained in the volume expansion of the minimal surface which is asymptotic to Σ when the minimal surface approaches the conformal infinity. In the paper we give the explicit expression of Graham-Witten's conformal invariant for closed four dimensional submanifolds and find critical points of the conformal invariant in the case of Euclidean ambient spaces.

1. INTRODUCTION

In the introduction we give a description of the main result and some related background of the paper. The terminologies used in the introduction will be recalled in the next section.

Let (X^{d+1}, g_+) be a conformally compact Einstein manifold and $(M^d, [g_{confinf}])$ its conformal infinity. A given metric $\bar{g} \in [g_{confinf}]$ uniquely determines a special defining function r on a neighborhood of M in \bar{X} , upon to the conditions that $(r^2 g_+)|_M = \bar{g}$ and $|dr|_{r^2 g_+} = 1$ [12]. We denote $g_c = r^2 g_+$. With the special defining function r , one can identify $M \times [0, \epsilon)$, for some $\epsilon > 0$, with a neighborhood of M in \bar{X} . We denote the neighborhood by X_ϵ , and the identification

$$(1.1) \quad M \times [0, \epsilon) \cong X_\epsilon$$

is defined as follows: $(p, r) \in M \times [0, \epsilon)$ corresponds to the point obtained by following the flow of $\nabla^{g_c} r$ emanating from p for r units of time. g_c on $M \times [0, \epsilon)$ takes the form of

$$(1.2) \quad g_c = dr^2 + \tilde{g},$$

where \tilde{g} is a 1-parameter family of metrics on M with the parameter r . By solving the Einstein equation $Ric(g_+) = -dg_+$, for d odd the expansion of \tilde{g} is of the form

$$(1.3) \quad \tilde{g} = g^{(0)} + g^{(2)}r^2 + (\text{even powers}) + g^{(d-1)}r^{d-1} + g^{(d)}r^d + \dots$$

where $g^{(j)}$ are tensors on M and the dots stand for terms vanishing to higher order. For j even and $0 \leq j \leq d-1$, the tensor $g^{(j)}$ is locally formally determined by the boundary value $g^{(0)} = \bar{g}$, but $g^{(d)}$ is formally undetermined; for d even the expansion is

$$(1.4) \quad \tilde{g} = g^{(0)} + g^{(2)}r^2 + (\text{even powers}) + hr^d \log d + g^{(d)}r^d + \dots$$

where $g^{(j)}$ and h are locally formally determined for j even and $0 \leq j \leq d-2$ by $g^{(0)} = \bar{g}$.

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The main object of the paper is a minimal surface in the conformally compact Einstein manifold (X, g_+) with prescribed asymptotic boundary. Let Σ^n be a submanifold of M and $Y^{n+1} \hookrightarrow (X, g_+)$ be a minimal surface which is asymptotic to Σ . The problem of existence and regularity of such minimal surfaces has been studied by Anderson [3, 4], Hardt-Lin [27], Lin [30, 31, 32], Tonegawa [35], Han-Jiang [25] and Han-Shen-Wang [26]. We denote

$$(1.5) \quad g = \bar{g}|_{\Sigma}.$$

The connections with respect to (M, \bar{g}) and (Σ, g) will be denoted by $\bar{\nabla}$ and ∇ respectively, and the connection of the normal bundle $T^\perp \Sigma$ of the immersion $\Sigma \hookrightarrow (M^d, \bar{g})$ will be denoted by ∇^\perp .

Graham and Witten [16] have introduced a natural and useful way to reformulate Y . Namely, near the boundary M they express Y as a graph over $\Sigma \times [0, \epsilon)$ and expand the height functions of the graph in r . Near a point of Σ^n , let (x^i, y^α) be a local coordinate chart of M^d , where $1 \leq i \leq n$ and $n+1 \leq \alpha \leq d$, so that

$$(1.6) \quad \Sigma = \{y = 0\}; \quad \bar{g}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^\alpha}\right) = 0 \quad \text{on } \Sigma, \forall i, \alpha.$$

Note that via the identification (1.1), one has an extension of the coordinates (x^i, y^α) into X , which together with r forms a local coordinate chart of \bar{X} . The minimal surface Y can be written as a graph $\{y^\alpha = u^\alpha(x, r)\}$. That is, near the boundary $Y = (x^i, u^\alpha(x, r), r)$.

Graham and Witten [16] proved that for n odd

$$(1.7) \quad u = u^{(2)}r^2 + (\text{even powers}) + u^{(n+1)}r^{n+1} + u^{(n+2)}r^{n+2} + \dots,$$

and for n even

$$(1.8) \quad u = u^{(2)}r^2 + (\text{even powers}) + u^{(n)}r^n + w_n r^{n+2} \log r + u^{(n+2)}r^{n+2} + \dots,$$

where the $u^{(k)} = (u^{(k)\alpha})$, $k < n+2$, and $w_n = (w_n^\alpha)$ are functions of Σ and locally determined, while $u^{(n+2)}$ is not locally determined. They showed that the first non-vanishing coefficient $u^{(2)}$ is related to the mean curvature of $\Sigma \hookrightarrow (M, \bar{g})$ by

$$(1.9) \quad u^{(2)} = \frac{1}{2n}H.$$

In the paper, we calculate the coefficient $u^{(4)}$ for $n \geq 3$ by using the graphic minimal surface equation of Y .

Proposition 1.1. *For $n \geq 3$,*

$$(1.10) \quad \begin{aligned} 8n(n-2)u^{(4)} &= \Delta^\perp H^\alpha + g^{ij}g^{kl}\langle A_{ik}, H \rangle A_{jl}^\alpha - \frac{2}{n^2}|H|^2 H^\alpha \\ &+ g^{ij}\bar{g}^{\alpha\gamma}H^\beta \bar{W}_{i\beta j\gamma} + (n-4)\bar{g}^{\alpha\gamma}\bar{P}_{\beta\gamma}H^\beta - g^{ij}\bar{P}_{ij}H^\alpha + 2ng^{ik}g^{jl}\bar{P}_{kl}A_{ij}^\alpha \\ &+ n\bar{g}^{\alpha\gamma}g^{ij}(\bar{\nabla}_\gamma \bar{P}_{ij} - 2\bar{\nabla}_i \bar{P}_{j\gamma}) - \frac{n-2}{n}H^\beta H^\gamma \bar{\Gamma}_{\beta\gamma}^\alpha, \end{aligned}$$

where A_{ij} denotes the second fundamental form of $\Sigma \hookrightarrow (M, \bar{g})$, $\bar{\Gamma}$, \bar{P} and \bar{W} denote the Christoffel symbol, Schouten and Weyl tensor of \bar{g} respectively.

For $n = 2$, the calculation of w_2 in (1.8) can be carried out in a similar way. We have

$$(1.11) \quad w_2 = -\frac{1}{16}W,$$

where

$$(1.12) \quad \begin{aligned} W^\alpha &= \Delta^\perp H^\alpha + g^{ij}g^{kl}\langle A_{ik}, H \rangle A_{jl}^\alpha - \frac{1}{2}|H|^2 H^\alpha \\ &+ g^{ij}\bar{g}^{\alpha\gamma}H^\beta\bar{W}_{i\beta j\gamma} - 2\bar{g}^{\alpha\gamma}\bar{P}_{\beta\gamma}H^\beta - g^{ij}\bar{P}_{ij}H^\alpha + 4g^{ik}g^{jl}\bar{P}_{kl}A_{ij}^\alpha \\ &+ 2\bar{g}^{\alpha\gamma}g^{ij}(\bar{\nabla}_\gamma\bar{P}_{ij} - 2\bar{\nabla}_i\bar{P}_{j\gamma}). \end{aligned}$$

Note that if (X, g_+) is the Poicaré half-plane model of the hyperbolic space \mathbb{H}^{d+1} , for which $(M, \bar{g}) = \mathbb{R}^d$ and $g^{(2)} = 0$, then for Σ^2 we have

$$W = \Delta^\perp H + g^{ij}g^{kl}\langle A_{ik}, H \rangle A_{jl} - \frac{1}{2}|H|^2 H.$$

A surface $\Sigma^2 \hookrightarrow \mathbb{R}^d$ with $W = 0$ is well-known to be a Willmore surface. Han and Jiang [25] proved that in the case of $\Sigma^2 \in \mathbb{R}^3$ the log term vanishes if and only if Σ is a Willmore surface.

Our main aim of the paper is to calculate the conformal invariant, introduced by Graham and Witten (for each dimensional submanifold) [16], for closed four dimensional submanifolds. Here the conformal invariant for submanifolds is a functional of submanifolds which is invariant under conformal transformations of the metric of the ambient space. Recently, conformal invariants for hypersurfaces, constructed by using the volume renormalization of solutions to the singular Yamabe problem or general singular volume measures, and related aspects have been extensively studied [11, 14, 18, 19, 20, 21, 22, 36]. Graham-Witten's conformal invariants are obtained from the renormalization process of the volume of a minimal surface Y^{n+1} in a conformally compact Einstein manifold (X^{d+1}, g_+) which is asymptotic to a submanifold Σ^n immersed in the conformal infinity of X .

Near the asymptotic boundary, the volume form of Y takes the form of

$$(1.13) \quad d\mu_Y = r^{-n-1}[v^{(0)} + v^{(2)}r^2 + (\text{even powers}) + v^{(n)}r^n + \dots]d\mu_\Sigma dr,$$

where $v^{(j)}$ are locally determined functions of Σ , $v^{(n)} = 0$ for n odd, and $d\mu_\Sigma$ is the volume form of (Σ, g) . As $\epsilon \rightarrow 0$ and for n odd, the volume

$$(1.14) \quad Vol_{g_+}(Y \cap \{r > \epsilon\}) = c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + (\text{odd powers}) + c_{n-1}\epsilon^{-1} + c_n + o(1),$$

and for n even

$$(1.15) \quad Vol_{g_+}(Y \cap \{r > \epsilon\}) = c_0\epsilon^{-n} + (\text{even powers}) + c_{n-2}\epsilon^{-2} + L_n \log \frac{1}{\epsilon} + c_n + o(1),$$

where

$$(1.16) \quad L_n = \int_\Sigma v^{(n)} d\mu_\Sigma.$$

An area law was proposed by Ryu and Takayanagi [33, 34] which holographically identifies the volume in (1.14) or (1.15) with the entanglement entropy in quantum (conformal) field theories. Graham and Witten proved the following

Theorem 1.2. ([16]) *If n is odd, then c_n is independent of the choice of special defining function. If n is even, then L_n is independent of the choice of special defining function.*

Note that there is the one-to-one correspondence between representative metrics of the conformal class $(M, [g_{confinf}])$ and special defining functions. Hence c_n for n odd and L_n for n even are conformal invariants. c_n is called the renormalized volume of Y . For $n = 1$, an explicit expression of the renormalized volume c_1 has been obtained by Alexakis and Mazzeo [2]. For $n = 2$, Graham and Witten [16] showed that

$$-8L_2 = \int_{\Sigma} (|H|^2 + 4g^{ij}\bar{P}_{ij})d\mu_{\Sigma},$$

where $\bar{P} = \frac{1}{d-2}(\bar{Ric} - \frac{1}{2(d-1)}\bar{R}\bar{g})$ is the Schouten tensor of (M^d, \bar{g}) . L_2 is closely related to the Willmore functional. For example, when Σ is a closed two dimensional hypersurface in (M^3, \bar{g}) , it easily follows from the Gauss formula and Gauss-Bonnet formula that

$$-8L_2 = 8\pi\chi(\Sigma) + 2 \int_{\Sigma} |\overset{\circ}{A}|^2 d\mu_{\Sigma},$$

where $\overset{\circ}{A}$ denotes the traceless part of the second fundamental form of Σ .

Volume renormalization of a conformally compact Einstein manifold had been studied at almost the same time [28, 29, 12], which can be viewed as the extreme case under the setting of the paper: $\Sigma = M$ and $Y = X$. The log terms of the volume expansion were calculated explicitly in lower dimensions. For example, in dimension two

$$L_{d=2} = -\frac{1}{4} \int_{M^2} R_{\bar{g}} d\mu_{\bar{g}},$$

and in dimension four

$$(1.17) \quad L_{d=4} = \frac{1}{4} \int_{M^4} \sigma_2(\bar{P}) d\mu_{\bar{g}},$$

where \bar{P} is the Schouten tensor of \bar{g} and σ_2 is the second elementary symmetric function. The geometry and topology of closed four dimensional manifolds which admit a metric \bar{g} such that $\int_{M^4} \bar{R} d\mu_{\bar{g}} > 0$ and $\int_{M^4} \sigma_2(\bar{P}) d\mu_{\bar{g}} > 0$ has been studied by Gursky [23] and Chang-Gursky-Yang [7]. For discussions on the renormalized volume of conformally compact Einstein manifold, see for instance [1, 5, 6, 8, 10, 17].

In the paper, we calculate L_4 for closed four dimensional submanifolds.

Proposition 1.3. *Let Σ^4 be a closed submanifold in (M^d, \bar{g}) , we have*

$$(1.18) \quad \begin{aligned} 128L_4 &= \int_{\Sigma} (|\nabla^{\perp} H|^2 - g^{ik}g^{jl}\langle A_{ij}, H \rangle \langle A_{kl}, H \rangle + \frac{7}{16}|H|^4) d\mu_g \\ &+ 16 \int_{\Sigma} [(g^{ij}\bar{P}_{ij})^2 - g^{ik}g^{jl}\bar{P}_{ij}\bar{P}_{kl} + g^{ij}\bar{g}^{\alpha\beta}\bar{P}_{i\alpha}\bar{P}_{j\beta} - \frac{1}{d-4}g^{ij}\bar{B}_{ij}] d\mu_g \\ &+ \int_{\Sigma} (-16g^{ik}g^{jl}\bar{P}_{ij}\langle A_{kl}, H \rangle + 5g^{ij}\bar{P}_{ij}|H|^2 + 8\bar{P}(H, H) - g^{ij}\bar{W}_{i\alpha j\beta}H^{\alpha}H^{\beta}) d\mu_g \\ &- 8 \int_{\Sigma} g^{ij}H^{\beta}(\bar{\nabla}_{\beta}\bar{P}_{ij} - 2\bar{\nabla}_j\bar{P}_{i\beta}) d\mu_g, \end{aligned}$$

where \bar{B}_{ij} is the Bach tensor of (M^d, \bar{g}) .

When (X^{d+1}, g_+) is the Poincaré half-plane model with $(M^d, \bar{g}) = \mathbb{R}^d$, L_4 is simplified to

$$(1.19) \quad \frac{1}{128} \int_{\Sigma} (|\nabla^\perp H|^2 - g^{ik} g^{jl} \langle A_{ij}, H \rangle \langle A_{kl}, H \rangle + \frac{7}{16} |H|^4) d\mu_g.$$

In the last part of the paper, we will consider the functional

$$(1.20) \quad \mathcal{L}_4(\Sigma^4) := \frac{1}{2} \int_{\Sigma} (|\nabla^\perp H|^2 - g^{ik} g^{jl} \langle A_{ij}, H \rangle \langle A_{kl}, H \rangle + \frac{7}{16} |H|^4) d\mu_g,$$

for closed four dimensional submanifolds $\Sigma^4 \hookrightarrow \mathbb{R}^d$. Guven [24] has constructed a bending energy for closed four dimensional submanifolds immersed in \mathbb{R}^5 , which is invariant under special conformal transformations of \mathbb{R}^5 and reads

$$(1.21) \quad H_2 = \frac{1}{2} \int_{\Sigma} (|\nabla H|^2 + 2|A|^2 H^2 - \frac{7}{8} |H|^4) d\mu_g.$$

Although \mathcal{L}_4 is not conformally invariant, the Euler-Lagrange equation of \mathcal{L}_4 is identically the one of L_4 if we fix the ambient space as the Euclidean space \mathbb{R}^d . It was pointed out by Graham in [14] that he and N. Reichert prove that the submanifold variation of L_n is related to w_n in (1.8).

In general, the functional \mathcal{L}_4 is unbounded from below and above even for closed four dimensional submanifolds with a fixed topology. Let $\mathbb{S}^k(r)$ denote the round sphere in \mathbb{R}^{k+1} of radius r . Examples of critical points of \mathcal{L}_4 include \mathbb{S}^4 , $\mathbb{S}^3(1) \times \mathbb{S}^1(\frac{1}{\sqrt{3}})$, $\mathbb{S}^3(1) \times \mathbb{S}^1(\sqrt{\frac{3}{5}})$, $\mathbb{S}^2(1) \times \mathbb{S}^2(1)$, $\mathbb{S}^2(1) \times \mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^1(\frac{1}{\sqrt{2}})$, $\mathbb{S}^2(1) \times \mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^1(\frac{3}{\sqrt{10}})$, $\mathbb{S}^1(1) \times \mathbb{S}^1(1) \times \mathbb{S}^1(1) \times \mathbb{S}^1(1)$, and $\mathbb{S}^1(1) \times \mathbb{S}^1(1) \times \mathbb{S}^1(1) \times \mathbb{S}^1(\frac{3}{\sqrt{5}})$. These are all the closed and four dimensional critical points of \mathcal{L}_4 which is a product of round spheres.

The calculations in the paper are elementary. In Section 2, we give a brief recall of conformally compact Einstein manifolds and minimal surfaces in conformally compact Einstein manifolds. We identify the constituent components of $u^{(4)}$ for $n \geq 3$ by using the graphic minimal surface equation. In Section 3, we reformulate $u^{(4)}$ in a covariant form and prove Proposition 1.1. In Section 4, we calculate the coefficient $v^{(4)}$ in (1.13) for $n \geq 4$, from which Proposition 1.3 follows so that we obtain Graham-Witten's conformal invariant \mathcal{L}_4 for closed four dimensional submanifolds. In the last section, we consider Graham-Witten's conformal invariant for closed four dimensional submanifolds of Euclidean spaces and find simple critical points of it.

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Notice. After the completion of the manuscript, we learn that, along with other interesting results and discussions, $u^{(4)}$ and L_4 have also been calculated by Graham and Reichert [15]. The critical points contained in the paper have also been found in [15]. They point out that Guven's functional H_2 in (1.21) should actually be \mathcal{L}_4 in case of $(M^d, \bar{g}) = \mathbb{R}^5$.

2. MINIMAL SUBMANIFOLDS IN CONFORMALLY COMPACT EINSTEIN MANIFOLDS

In the first part of this section, referring mainly to [12, 16], we give a brief review to the related aspects of conformally compact Einstein manifolds, and minimal submanifolds in conformally compact Einstein manifolds. In the second part, we identify the constituent components of $u^{(4)}$ from the graphic minimal surface equation of Y .

2.1. Minimal surfaces in a conformally compact Einstein manifold. Let \bar{X} be an $(d+1)$ -dimensional manifold with a d -dimensional boundary M . We call a function $r \in C^\infty(\bar{X})$ a defining function for M if it satisfies that $r|_X > 0$, $r|_M = 0$ and $(dr)|_M \neq 0$. A metric g_+ defined on X is said to be conformally compact if $r^2 g_+$ extends to a metric on \bar{X} . Given a conformally compact manifold (X, g_+) , $(r^2 g_+)|_{TM}$ induces a conformal class of metrics as r runs over the space of defining functions, called the conformal infinity of (X, g_+) and denoted by $(M, [g_{confinf}])$. Let $g_c = r^2 g_+$. The sectional curvature of (X, g_+) is asymptotic to $-(|dr|_{g_c}^2)|_M$, which is independent of the choices of r , as $r \rightarrow 0$.

If a conformally compact manifold (X, g_+) satisfies $Ric_{g_+} = -dg_+$, (X, g_+) is called a conformally compact Einstein manifold. For a conformally compact Einstein metric and any defining function r , $|dr|_{g_c}^2 = 1$ on M . Hence conformally compact Einstein manifolds are asymptotic hyperbolic Einstein manifolds. From now on let (X^{d+1}, g_+) be a conformally compact Einstein manifold. For a conformally compact Einstein manifold (X, g_+) , as indicated in the introduction, there is a one-to-one correspondence between $\bar{g} \in [g_{confinf}]$ and special defining functions defined near ∂X , and the identification (1.1) of $M \times [0, \epsilon)$ with a neighborhood of M in \bar{X} so that $g_+ = r^{-2}(dr^2 + \tilde{g})$, where an expansion of \tilde{g} is given by (1.3) and (1.4) for d odd and even respectively. The determined coefficients in these expansions can be calculated and are combinations of \bar{g} and its derivatives, see [9, 13]. For example, for $d \geq 3$ one has

$$(2.1) \quad g^{(2)} = -\bar{P} = -\frac{1}{d-2}[\bar{Ric} - \frac{\bar{R}}{2(d-1)}\bar{g}],$$

here \bar{P} is the Schouten tensor of \bar{g} . It is well-known that

$$(2.2) \quad \bar{R}_{ABCD} = \bar{W}_{ABCD} + \bar{P}_{AC}\bar{g}_{BD} + \bar{P}_{BD}\bar{g}_{AC} - \bar{P}_{AD}\bar{g}_{BC} - \bar{P}_{BC}\bar{g}_{AD},$$

here \bar{W} is the Weyl tensor of \bar{g} . For $d \geq 5$,

$$(2.3) \quad g_{CD}^{(4)} = \frac{1}{4(4-d)}\bar{B}_{CD} + \frac{1}{4}\bar{g}^{EF}\bar{P}_{CE}\bar{P}_{DF},$$

where

$$(2.4) \quad \bar{B}_{CD} = \bar{\Delta}\bar{P}_{CD} - \bar{\nabla}^E\bar{\nabla}_D\bar{P}_{CE} + \bar{P}^{EF}\bar{W}_{CEDF}$$

is the Bach tensor of \bar{g} .

Let (X^{d+1}, g_+) be a conformally compact Einstein manifold and r a special defining function such that $g_c|_M = \bar{g}$. Now we consider a minimal surface Y^{n+1} , $0 \leq n \leq d-1$, of (X, g_+) with the smooth boundary $\Sigma^n \subset M^d$. Near a point of Σ , let (x^i, y^α) be a local coordinate chart as indicated in (1.6), which is extended into a neighborhood of the point in \bar{X} via the identification (1.1). The second fundamental form of $\Sigma \hookrightarrow (M, \bar{g})$ is

$$(2.5) \quad A_{ij}^\alpha = \bar{g}^{\alpha\beta}\bar{g}(\bar{\nabla}_i\partial_j, \partial_\beta) = -\frac{1}{2}\bar{g}^{\alpha\beta}\partial_\beta\bar{g}_{ij} = \bar{\Gamma}_{ij}^\alpha.$$

We consider minimal surfaces Y in (X, g_+) which may be written as a graph $\{y^\alpha = u^\alpha(x, r)\}$. That is near the boundary, $Y = (x^i, u^\alpha(x, r), r)$. Let

$$(2.6) \quad h = g_+|_Y, \quad \bar{h} = r^2 h = g_c|_Y = (dr^2 + \tilde{g})|_Y,$$

where \tilde{g} is given by (1.2). Near the boundary, the tangent space of Y is spanned by

$$Y_r = \partial_r + u_r^\gamma \partial_\gamma,$$

and

$$Y_i = \partial_i + u_i^\gamma \partial_\gamma, \quad i = 1, 2, \dots, n.$$

Then

$$\begin{aligned} \bar{h}_{rr} &= \bar{h}(Y_r, Y_r) = 1 + \tilde{g}_{\alpha\beta} u_r^\alpha u_r^\beta, \\ \bar{h}_{ir} &= \bar{h}(Y_i, Y_r) = \tilde{g}_{i\alpha} u_r^\alpha + \tilde{g}_{\alpha\beta} u_i^\alpha u_r^\beta, \end{aligned}$$

and

$$\bar{h}_{ij} = \bar{h}(Y_i, Y_j) = \tilde{g}_{ij} + \tilde{g}_{i\alpha} u_j^\alpha + \tilde{g}_{j\alpha} u_i^\alpha + \tilde{g}_{\alpha\beta} u_i^\alpha u_j^\beta.$$

Graham and Witten [16] showed that the graphic minimal surface equation of Y is

$$M(u) = 0,$$

where

$$\begin{aligned} M(u)_\gamma &= [r\partial_r - (n+1) + \frac{1}{2}r\partial_r \bar{L}][\bar{h}^{rr} \tilde{g}_{\beta\gamma} u_r^\beta + \bar{h}^{ir}(\tilde{g}_{i\gamma} + \tilde{g}_{\beta\gamma} u_i^\beta)] \\ &\quad + r[\partial_j + \frac{1}{2}\partial_j \bar{L}][\bar{h}^{rj} \tilde{g}_{\beta\gamma} u_r^\beta + \bar{h}^{ij}(\tilde{g}_{i\gamma} + \tilde{g}_{\beta\gamma} u_i^\beta)] \\ &\quad - \frac{1}{2}r\bar{h}^{ij}[\partial_\gamma \tilde{g}_{ij} + 2\partial_\gamma \tilde{g}_{i\alpha} u_j^\alpha + \partial_\gamma \tilde{g}_{\alpha\beta} u_i^\alpha u_j^\beta] \\ &\quad - r\bar{h}^{ir}[\partial_\gamma \tilde{g}_{i\alpha} u_r^\alpha + \partial_\gamma \tilde{g}_{\alpha\beta} u_i^\alpha u_r^\beta] \\ &\quad - \frac{1}{2}r\bar{h}^{rr} \partial_\gamma \tilde{g}_{\alpha\beta} u_r^\alpha u_r^\beta, \end{aligned} \quad (2.7)$$

and

$$(2.8) \quad \bar{L} = \log \det \bar{h}.$$

By the choice of the local coordinates (x^i, y^α) on M , we have $u(x, 0) = 0$. Graham and Witten [16] found that at $r = 0$,

$$(2.9) \quad u_r = 0,$$

which means that Y intersects with M orthogonally; and for $n \geq 1$

$$(2.10) \quad u_{rr}^\alpha = -\frac{1}{2n} g^{ij} \bar{g}^{\alpha\beta} \partial_\beta \bar{g}_{ij} = \frac{1}{n} H^\alpha,$$

here H is the mean curvature of $\Sigma \hookrightarrow (M, \bar{g})$. The expansion of the height functions u in r , up to a critical order, takes the form of (1.7) for n odd and of (1.8) for n even. Namely, for n odd

$$u = u^{(2)} r^2 + (\text{even powers}) + u^{(n+1)} r^{n+1} + u^{(n+2)} r^{n+2} + \dots,$$

and for n even

$$u = u^{(2)} r^2 + (\text{even powers}) + u^{(n)} r^n + w_n r^{n+2} \log r + u^{(n+2)} r^{n+2} + \dots,$$

where $u^{(k)}, k < n+2$, and w_n are locally determined functions of Σ .

2.2. Coefficient $u^{(4)}$ in (1.7) and (1.8) for $n \geq 3$. For $n \geq 3$, let

$$(2.11) \quad u = vr^2 + wr^4 + O(r^5),$$

where, it follows from (2.10),

$$(2.12) \quad v^\alpha = \frac{1}{2n} H^\alpha.$$

It follows from (2.8) that

$$\partial_r L = O(r).$$

Note that

$$\begin{aligned} \bar{h}_{ir} &= O(r^3), \quad \bar{h}^{ir} = O(r^3), \\ \tilde{g}_{i\alpha} &= O(r^2), \quad u_r^\beta = O(r), \quad u_i^\beta = O(r^2), \end{aligned}$$

and the fact that the coefficient w in (2.11) can be derived by tracing the coefficient of r^3 in (2.7). First, we have

$$(2.13) \quad M(u)_\gamma = [r\partial_r - (k+1)](\bar{h}^{rr}\tilde{g}_{\beta\gamma}u_r^\beta) + \frac{1}{2}r\partial_r\bar{L}\bar{h}^{rr}\tilde{g}_{\beta\gamma}u_r^\beta$$

$$(2.14) \quad + r\partial_j[\bar{h}^{ij}(\tilde{g}_{i\gamma} + \tilde{g}_{\beta\gamma}u_i^\beta)] + \frac{1}{2}r\partial_j\bar{L}\bar{h}^{ij}(\tilde{g}_{i\gamma} + \tilde{g}_{\beta\gamma}u_i^\beta)$$

$$(2.15) \quad -\frac{1}{2}r\bar{h}^{ij}\partial_\gamma\tilde{g}_{ij} - r\bar{h}^{ij}\partial_\gamma\tilde{g}_{i\alpha}u_j^\alpha - \frac{1}{2}r\bar{h}^{rr}\partial_\gamma\tilde{g}_{\alpha\beta}u_r^\alpha u_r^\beta + o(r^3).$$

Lemma 2.1. *For the right hand side of (2.13), we have*

$$(2.16) \quad \begin{aligned} &[r\partial_r - (n+1)](\bar{h}^{rr}\tilde{g}_{\beta\gamma}u_r^\beta) + \frac{1}{2}r\partial_r\bar{L}\bar{h}^{rr}\tilde{g}_{\beta\gamma}u_r^\beta \\ &= -2nr\bar{g}_{\beta\gamma}v^\beta - (n-2)(2g_{\beta\gamma}^{(2)}v^\beta + 2\partial_\alpha\bar{g}_{\beta\gamma}v^\alpha v^\beta + 4\bar{g}_{\beta\gamma}w^\beta)r^3 \\ &\quad + r^3[-8|v|^2\bar{g}_{\beta\gamma}v^\beta + 2g^{ij}g_{ij}^{(2)}\bar{g}_{\beta\gamma}v^\beta] + o(r^3). \end{aligned}$$

Proof. Note that

$$\bar{h}^{rr} = 1 - 4|v|^2r^2 + o(r^2).$$

Hence

$$\begin{aligned} \bar{h}^{rr}\tilde{g}_{\beta\gamma}u_r^\beta &= (1 - 4|v|^2r^2)(\bar{g}_{\beta\gamma} + r^2g_{\beta\gamma}^{(2)} + r^2\partial_\alpha\bar{g}_{\beta\gamma}v^\alpha)(2rv^\beta + 4r^3w^\beta) + o(r^3) \\ &= 2r\bar{g}_{\beta\gamma}v^\beta + (-8|v|^2\bar{g}_{\beta\gamma}v^\beta + 2g_{\beta\gamma}^{(2)}v^\beta + 2\partial_\alpha\bar{g}_{\beta\gamma}v^\alpha v^\beta + 4\bar{g}_{\beta\gamma}w^\beta)r^3 + o(r^3). \end{aligned}$$

Therefore,

$$(2.17) \quad \begin{aligned} &[r\partial_r - (n+1)](\bar{h}^{rr}\tilde{g}_{\beta\gamma}u_r^\beta) = -2nr\bar{g}_{\beta\gamma}v^\beta \\ &-(n-2)(-8|v|^2\bar{g}_{\beta\gamma}v^\beta + 2g_{\beta\gamma}^{(2)}v^\beta + 2\partial_\alpha\bar{g}_{\beta\gamma}v^\alpha v^\beta + 4\bar{g}_{\beta\gamma}w^\beta)r^3 + o(r^3). \end{aligned}$$

Note that

$$\begin{aligned} \partial_r\bar{L} &= \bar{h}^{rr}\partial_r\bar{h}_{rr} + \bar{h}^{ij}\partial_r\bar{h}_{ij} + o(r) \\ &= 8r\bar{g}_{\alpha\beta}v^\alpha v^\beta + g^{ij}(2rg_{ij}^{(2)} + 2rg^{ij}\partial_\alpha\bar{g}_{ij}v^\alpha) + o(r) \\ &= r[(8-8n)|v|^2 + 2g^{ij}g_{ij}^{(2)}] + o(r), \end{aligned}$$

where in the last equality we used (2.5) and (2.12). Hence

$$(2.18) \quad \frac{1}{2}r\partial_r\overline{Lh}^{rr}\tilde{g}_{\beta\gamma}u_r^\beta = r^3[(8-8n)|v|^2 + 2g^{ij}g_{ij}^{(2)}]\overline{g}_{\beta\gamma}v^\beta + o(r^3).$$

(2.16) then follows from (2.17) and (2.18). \square

Lemma 2.2. *For (2.14), we have*

$$(2.19) \quad \begin{aligned} & r\partial_j[\overline{h}^{ij}(\tilde{g}_{i\gamma} + \tilde{g}_{\beta\gamma}u_i^\beta)] + \frac{1}{2}r\partial_j\overline{Lh}^{ij}(\tilde{g}_{i\gamma} + \tilde{g}_{\beta\gamma}u_i^\beta) \\ &= r^3g^{ij}[\partial_jg_{i\gamma}^{(2)} + \partial_j\partial_\beta\overline{g}_{i\gamma}v^\beta + \partial_\beta\overline{g}_{i\gamma}v_j^\beta + \partial_j\overline{g}_{\beta\gamma}v_i^\beta + \overline{g}_{\beta\gamma}\partial_jv_i^\beta] \\ & \quad - r^3g^{kl}\Gamma_{kl}^i(g_{i\gamma}^{(2)} + \partial_\beta\overline{g}_{i\gamma}v^\beta + \overline{g}_{\beta\gamma}v_i^\beta) + o(r^3). \end{aligned}$$

Proof. It is easy to see that

$$\begin{aligned} r\partial_j[\overline{h}^{ij}(\tilde{g}_{i\gamma} + \tilde{g}_{\beta\gamma}u_i^\beta)] &= -r^3g^{ik}g^{jl}\partial_jg_{kl}(g_{i\gamma}^{(2)} + \partial_\beta\overline{g}_{i\gamma}v^\beta + \overline{g}_{\beta\gamma}v_i^\beta) \\ & \quad + r^3g^{ij}\partial_j\tilde{g}_{i\gamma} + r^3g^{ij}(\partial_j\overline{g}_{\beta\gamma}v_i^\beta + \overline{g}_{\beta\gamma}\partial_jv_i^\beta) + o(r^3). \end{aligned}$$

It then follows from

$$\partial_j\tilde{g}_{i\gamma} = r^2(\partial_jg_{i\gamma}^{(2)} + \partial_j\partial_\beta\overline{g}_{i\gamma}v^\beta + \partial_\beta\overline{g}_{i\gamma}v_j^\beta) + o(r^2)$$

that

$$(2.20) \quad \begin{aligned} & r\partial_j[\overline{h}^{ij}(\tilde{g}_{i\gamma} + \tilde{g}_{\beta\gamma}u_i^\beta)] = -r^3g^{ik}g^{jl}\partial_jg_{kl}(g_{i\gamma}^{(2)} + \partial_\beta\overline{g}_{i\gamma}v^\beta + \overline{g}_{\beta\gamma}v_i^\beta) \\ & \quad + r^3g^{ij}[\partial_jg_{i\gamma}^{(2)} + \partial_j\partial_\beta\overline{g}_{i\gamma}v^\beta + \partial_\beta\overline{g}_{i\gamma}v_j^\beta + \partial_j\overline{g}_{\beta\gamma}v_i^\beta + \overline{g}_{\beta\gamma}\partial_jv_i^\beta] + o(r^3). \end{aligned}$$

It is easy to see that

$$(2.21) \quad \begin{aligned} & \frac{1}{2}r\partial_j\overline{Lh}^{ij}(\tilde{g}_{i\gamma} + \tilde{g}_{\beta\gamma}u_i^\beta) \\ &= \frac{1}{2}r^3g^{ij}g^{kl}\partial_jg_{kl}(g_{i\gamma}^{(2)} + \partial_\beta\overline{g}_{i\gamma}v^\beta + \overline{g}_{\beta\gamma}v_i^\beta) + o(r^3). \end{aligned}$$

(2.19) then follows from (2.20) and (2.21). \square

Lemma 2.3. *For (2.15), we have*

$$(2.22) \quad \begin{aligned} & -\frac{1}{2}r\overline{h}^{ij}\partial_\gamma\tilde{g}_{ij} - r\overline{h}^{ij}\partial_\gamma\tilde{g}_{i\alpha}u_j^\alpha - \frac{1}{2}r\overline{h}^{rr}\partial_\gamma\tilde{g}_{\alpha\beta}u_r^\alpha u_r^\beta \\ &= 2nr\overline{g}_{\beta\gamma}v^\beta + [-\frac{1}{2}g^{ij}\partial_\gamma g_{ij}^{(2)} - \frac{1}{2}g^{ij}\partial_\gamma\partial_\beta\overline{g}_{ij}v^\beta - g^{ik}g^{jl}g_{kl}^{(2)}A_{ij}^\beta\overline{g}_{\beta\gamma}]r^3 \\ & \quad + 2r^3g^{ik}g^{jl}\langle A_{kl}, v \rangle A_{ij}^\beta\overline{g}_{\beta\gamma} + r^3[-g^{ij}\partial_\gamma\overline{g}_{i\alpha}v_j^\alpha - 2\partial_\gamma\overline{g}_{\alpha\beta}v^\alpha v^\beta] + o(r^3). \end{aligned}$$

Proof. It is clear that we have

$$\overline{h}^{ij} = g^{ij} - g^{ik}g^{jl}(g_{kl}^{(2)} + \partial_\alpha\overline{g}_{kl}v^\alpha)r^2 + o(r^2),$$

and

$$\partial_\gamma\tilde{g}_{ij} = \partial_\gamma\overline{g}_{ij} + r^2(\partial_\gamma g_{ij}^{(2)} + \partial_\gamma\partial_\beta\overline{g}_{ij}v^\beta) + o(r^2).$$

Hence by using (2.5), we get

$$\begin{aligned}
(2.23) \quad & -\frac{1}{2}r\bar{h}^{ij}\partial_\gamma\tilde{g}_{ij} = 2nr\bar{g}_{\beta\gamma}v^\beta \\
& +[-\frac{1}{2}g^{ij}\partial_\gamma g_{ij}^{(2)} - \frac{1}{2}g^{ij}\partial_\gamma\partial_\beta\bar{g}_{ij}v^\beta - g^{ik}g^{jl}g_{kl}^{(2)}A_{ij}^\beta\bar{g}_{\beta\gamma}]r^3 \\
& +2r^3g^{ik}g^{jl}\langle A_{kl}, v \rangle A_{ij}^\beta\bar{g}_{\beta\gamma} + o(r^3).
\end{aligned}$$

On the other hand, it's easy to see that

$$\begin{aligned}
(2.24) \quad & -r\bar{h}^{ij}\partial_\gamma\tilde{g}_{i\alpha}u_j^\alpha - \frac{1}{2}r\bar{h}^{rr}\partial_\gamma\tilde{g}_{\alpha\beta}u_r^\alpha u_r^\beta \\
& = r^3[-g^{ij}\partial_\gamma\bar{g}_{i\alpha}v_j^\alpha - 2\partial_\gamma\bar{g}_{\alpha\beta}v^\alpha v^\beta] + o(r^3).
\end{aligned}$$

(2.22) then follows from (2.23) and (2.24). \square

Putting (2.16) (2.19) and (2.22) together, we get the following

Corollary 2.1. *For $n \geq 3$, we have*

$$\begin{aligned}
(2.25) \quad M(u)_\gamma &= -(n-2)(2g_{\beta\gamma}^{(2)}v^\beta + 2\partial_\alpha\bar{g}_{\beta\gamma}v^\alpha v^\beta + 4\bar{g}_{\beta\gamma}v^\beta)r^3 \\
&+ r^3[-8|v|^2\bar{g}_{\beta\gamma}v^\beta + 2g^{ij}g_{ij}^{(2)}\bar{g}_{\beta\gamma}v^\beta] \\
&+ r^3g^{ij}[\partial_j g_{i\gamma}^{(2)} + \partial_j\partial_\beta\bar{g}_{i\gamma}v^\beta + \partial_\beta\bar{g}_{i\gamma}v_j^\beta + \partial_j\bar{g}_{\beta\gamma}v_i^\beta + \bar{g}_{\beta\gamma}\partial_j v_i^\beta] \\
&- r^3g^{kl}\Gamma_{kl}^i(g_{i\gamma}^{(2)} + \partial_\beta\bar{g}_{i\gamma}v^\beta + \bar{g}_{\beta\gamma}v_i^\beta) \\
&+ [-\frac{1}{2}g^{ij}\partial_\gamma g_{ij}^{(2)} - \frac{1}{2}g^{ij}\partial_\gamma\partial_\beta\bar{g}_{ij}v^\beta - g^{ik}g^{jl}g_{kl}^{(2)}A_{ij}^\beta\bar{g}_{\beta\gamma}]r^3 \\
&+ 2r^3g^{ik}g^{jl}\langle A_{kl}, v \rangle A_{ij}^\beta\bar{g}_{\beta\gamma} + r^3[-g^{ij}\partial_\gamma\bar{g}_{i\alpha}v_j^\alpha - 2\partial_\gamma\bar{g}_{\alpha\beta}v^\alpha v^\beta] + o(r^3).
\end{aligned}$$

Note that

$$w = u^{(4)},$$

so we have

Proposition 2.4. *Let Y^{n+1} , $n \geq 3$, be a minimal surface in (X^{d+1}, g_+) . Then*

$$\begin{aligned}
(2.26) \quad 4(n-2)\bar{g}_{\beta\gamma}u^{(4)\beta} &= -2(n-2)g_{\beta\gamma}^{(2)}v^\beta - 8|v|^2\bar{g}_{\beta\gamma}v^\beta + 2g^{ij}g_{ij}^{(2)}\bar{g}_{\beta\gamma}v^\beta \\
&- g^{ik}g^{jl}g_{kl}^{(2)}A_{ij}^\beta\bar{g}_{\beta\gamma} + 2g^{ik}g^{jl}\langle A_{kl}, v \rangle A_{ij}^\beta\bar{g}_{\beta\gamma} \\
&+ Q_\gamma,
\end{aligned}$$

where

$$\begin{aligned}
(2.27) \quad Q_\gamma &= g^{ij}[\partial_j g_{i\gamma}^{(2)} + \partial_j\partial_\beta\bar{g}_{i\gamma}v^\beta + \partial_\beta\bar{g}_{i\gamma}v_j^\beta + \partial_j\bar{g}_{\beta\gamma}v_i^\beta - \partial_\gamma\bar{g}_{i\alpha}v_j^\alpha + \bar{g}_{\beta\gamma}\partial_j v_i^\beta] \\
&- g^{kl}\Gamma_{kl}^i(g_{i\gamma}^{(2)} + \partial_\beta\bar{g}_{i\gamma}v^\beta + \bar{g}_{\beta\gamma}v_i^\beta) \\
&- \frac{1}{2}g^{ij}\partial_\gamma g_{ij}^{(2)} - \frac{1}{2}g^{ij}\partial_\gamma\partial_\beta\bar{g}_{ij}v^\beta \\
&- 2(n-2)\partial_\alpha\bar{g}_{\beta\gamma}v^\alpha v^\beta - 2\partial_\gamma\bar{g}_{\alpha\beta}v^\alpha v^\beta.
\end{aligned}$$

3. PROOF OF PROPOSITION 1.1

Assume $n \geq 3$. In this section, we will reformulate Q_γ to get the expression of $u^{(4)}$, as given by (1.10). Let

$$Q^\alpha = \bar{g}^{\alpha\gamma} Q_\gamma.$$

It follows from (2.27) that

$$(3.1) \quad \begin{aligned} 4(n-2)u^{(4)\alpha} &= -2(n-2)\bar{g}^{\alpha\gamma}g_{\beta\gamma}^{(2)}v^\beta - 8|v|^2v^\alpha + 2g^{ij}g_{ij}^{(2)}v^\alpha \\ &\quad - g^{ik}g^{jl}g_{kl}^{(2)}A_{ij}^\alpha + 2g^{ik}g^{jl}\langle A_{kl}, v \rangle A_{ij}^\alpha + Q^\alpha, \end{aligned}$$

where

$$Q^\alpha = I^\alpha + II^\alpha,$$

and

$$(3.2) \quad \begin{aligned} I^\alpha &= g^{ij}[\partial_j v_i^\alpha - \frac{1}{2}\bar{g}^{\alpha\gamma}\partial_\gamma\partial_\beta\bar{g}_{ij}v^\beta + \bar{g}^{\alpha\gamma}\partial_j\partial_\beta\bar{g}_{i\gamma}v^\beta] + 2g^{ij}\bar{\Gamma}_{ij}^\alpha v_j^\beta \\ &\quad - 2(n-2)\bar{g}^{\alpha\gamma}\partial_\beta\bar{g}_{\eta\gamma}v^\beta v^\eta - 2\bar{g}^{\alpha\gamma}\partial_\gamma\bar{g}_{\beta\eta}v^\beta v^\eta \\ &\quad - \bar{g}^{\alpha\gamma}g^{kl}\Gamma_{kl}^i(\partial_\beta\bar{g}_{i\gamma}v^\beta + \bar{g}_{\beta\gamma}v_i^\beta), \\ II^\alpha &= g^{ij}\bar{g}^{\alpha\gamma}[-\frac{1}{2}\partial_\gamma g_{ij}^{(2)} + \partial_j g_{i\gamma}^{(2)} - \Gamma_{ij}^k g_{k\gamma}^{(2)}]. \end{aligned}$$

We first compute II^α .

Lemma 3.1. *We have*

$$(3.3) \quad II^\alpha = -\frac{1}{2}\bar{g}^{\alpha\gamma}g^{ij}(\bar{\nabla}_\gamma g_{ij}^{(2)} - 2\bar{\nabla}_i g_{j\gamma}^{(2)}) + \bar{g}^{\alpha\gamma}H^\beta g_{\beta\gamma}^{(2)}.$$

Proof. Let D range from 1 to d so that for $\partial_D = \partial_i$ for $D = i \leq n$ and $\partial_D = \partial_\alpha$ for $D = \alpha \geq n+1$. We have

$$(3.4) \quad \begin{aligned} &\partial_\gamma g_{ij}^{(2)} - \partial_j g_{i\gamma}^{(2)} - \partial_i g_{j\gamma}^{(2)} \\ &= (\bar{\nabla}_\gamma g_{ij}^{(2)} + \bar{\Gamma}_{\gamma i}^D g_{Dj}^{(2)} + \bar{\Gamma}_{\gamma j}^D g_{iD}^{(2)}) - (\bar{\nabla}_j g_{i\gamma}^{(2)} + \bar{\Gamma}_{ji}^D g_{D\gamma}^{(2)} + \bar{\Gamma}_{j\gamma}^D g_{iD}^{(2)}) \\ &\quad - (\bar{\nabla}_i g_{j\gamma}^{(2)} + \bar{\Gamma}_{ij}^D g_{D\gamma}^{(2)} + \bar{\Gamma}_{i\gamma}^D g_{jD}^{(2)}) \\ &= \bar{\nabla}_\gamma g_{ij}^{(2)} - \bar{\nabla}_j g_{i\gamma}^{(2)} - \bar{\nabla}_i g_{j\gamma}^{(2)} - 2\bar{\Gamma}_{ij}^D g_{D\gamma}^{(2)}. \end{aligned}$$

Hence by using (2.5), we get

$$\begin{aligned} II^\alpha &= g^{ij}\bar{g}^{\alpha\gamma}[-\frac{1}{2}\partial_\gamma g_{ij}^{(2)} + \partial_j g_{i\gamma}^{(2)} - \Gamma_{ij}^k g_{k\gamma}^{(2)}] \\ &= -\frac{1}{2}\bar{g}^{\alpha\gamma}g^{ij}(\bar{\nabla}_\gamma g_{ij}^{(2)} - 2\bar{\nabla}_i g_{j\gamma}^{(2)}) + \bar{g}^{\alpha\gamma}g^{ij}\bar{\Gamma}_{ij}^D g_{D\gamma}^{(2)} - \bar{g}^{\alpha\gamma}g^{ij}\Gamma_{ij}^k g_{k\gamma}^{(2)} \\ &= -\frac{1}{2}\bar{g}^{\alpha\gamma}g^{ij}(\bar{\nabla}_\gamma g_{ij}^{(2)} - 2\bar{\nabla}_i g_{j\gamma}^{(2)}) + \bar{g}^{\alpha\gamma}H^\beta g_{\beta\gamma}^{(2)}. \end{aligned}$$

□

We now deal with I^α .

Lemma 3.2. *We have*

$$(3.5) \quad \begin{aligned} \partial_j v_i^\alpha &= \nabla_j^\perp \nabla_i^\perp v^\alpha - g^{kl} \langle A_{ik}, v \rangle A_{jl}^\alpha - v_i^\beta \bar{\Gamma}_{j\beta}^\alpha - v_j^\beta \bar{\Gamma}_{i\beta}^\alpha + v_k^\alpha \Gamma_{ij}^k \\ &\quad - v^\beta (\partial_j \bar{\Gamma}_{i\beta}^\alpha + \bar{\Gamma}_{i\beta}^\gamma \bar{\Gamma}_{j\gamma}^\alpha + \bar{\Gamma}_{i\beta}^k \bar{\Gamma}_{jk}^\alpha - \Gamma_{ij}^k \bar{\Gamma}_{k\beta}^\alpha). \end{aligned}$$

Proof. Let $A, B, C, D = \{i, \alpha\} = 1, 2, \dots, d$. Let X^\perp denote the normal part of X and ∇^\perp the covariant differentiation with respect to the normal connection. We have

$$\begin{aligned} \bar{\nabla}_i v &= \partial_i v^\alpha \partial_\alpha + v^\beta \bar{\Gamma}_{i\beta}^A \partial_A \\ &= (v_i^\alpha + v^\beta \bar{\Gamma}_{i\beta}^\alpha) \partial_\alpha + v^\beta \bar{\Gamma}_{i\beta}^k \partial_k. \end{aligned}$$

Then

$$\begin{aligned} (\bar{\nabla}_j \bar{\nabla}_i v)^\perp &= [\bar{\nabla}_j (\bar{\nabla}_i v)]^\perp - \Gamma_{ij}^k \nabla_k^\perp v \\ &= \partial_j (v_i^\alpha + v^\beta \bar{\Gamma}_{i\beta}^\alpha) \partial_\alpha + v_i^\beta \bar{\Gamma}_{j\beta}^\alpha \partial_\alpha + v^\beta \bar{\Gamma}_{i\beta}^D \bar{\Gamma}_{Dj}^\alpha \partial_\alpha - \Gamma_{ij}^k \nabla_k^\perp v \\ &= (\partial_j v_i^\alpha + v_i^\beta \bar{\Gamma}_{j\beta}^\alpha + v_j^\beta \bar{\Gamma}_{i\beta}^\alpha - v_k^\alpha \Gamma_{ij}^k) \partial_\alpha \\ &\quad + v^\beta (\partial_j \bar{\Gamma}_{i\beta}^\alpha + \bar{\Gamma}_{i\beta}^D \bar{\Gamma}_{Dj}^\alpha - \Gamma_{ij}^k \bar{\Gamma}_{k\beta}^\alpha) \partial_\alpha. \end{aligned}$$

On the other hand,

$$\begin{aligned} \nabla_j^\perp \nabla_i^\perp v &= [\bar{\nabla}_j (\nabla_i^\perp v)]^\perp - \Gamma_{ij}^k \nabla_k^\perp v \\ &= [\bar{\nabla}_j (\bar{\nabla}_i v)]^\perp + g^{kl} \langle A_{ik}, v \rangle A_{jl} - \Gamma_{ij}^k \nabla_k^\perp v \\ &= (\bar{\nabla}_j \bar{\nabla}_i v)^\perp + g^{kl} \langle A_{ik}, v \rangle A_{jl}. \end{aligned}$$

Therefore, (3.5) follows from

$$\begin{aligned} \nabla_j^\perp \nabla_i^\perp v &= (\partial_j v_i^\alpha + v_i^\beta \bar{\Gamma}_{j\beta}^\alpha + v_j^\beta \bar{\Gamma}_{i\beta}^\alpha - v_k^\alpha \Gamma_{ij}^k) \partial_\alpha \\ &\quad + v^\beta (\partial_j \bar{\Gamma}_{i\beta}^\alpha + \bar{\Gamma}_{i\beta}^D \bar{\Gamma}_{Dj}^\alpha - \Gamma_{ij}^k \bar{\Gamma}_{k\beta}^\alpha) \partial_\alpha \\ &\quad + g^{kl} \langle A_{ik}, v \rangle A_{jl}. \end{aligned}$$

□

Lemma 3.3. *We have*

$$(3.6) \quad \begin{aligned} &g^{ij} [-\frac{1}{2} \bar{g}^{\alpha\gamma} \partial_\gamma \partial_\beta \bar{g}_{ij} v^\beta + \bar{g}^{\alpha\gamma} \partial_j \partial_\beta \bar{g}_{i\gamma} v^\beta] \\ &= g^{ij} v^\beta \partial_\beta \bar{\Gamma}_{ij}^\alpha + g^{ij} \Gamma_{ij}^l \bar{g}^{\alpha\xi} v^\beta \partial_\beta \bar{g}_{l\xi} + 2n \bar{g}^{\alpha\xi} v^\beta v^\eta \partial_\beta \bar{g}_{\xi\eta}. \end{aligned}$$

Proof. Note that

$$\bar{\Gamma}_{ij}^\alpha = \frac{1}{2} \bar{g}^{\alpha\gamma} (\partial_i \bar{g}_{j\gamma} + \partial_j \bar{g}_{i\gamma} - \partial_\gamma \bar{g}_{ij}) + \frac{1}{2} \bar{g}^{\alpha k} (\partial_i \bar{g}_{jk} + \partial_j \bar{g}_{ik} - \partial_k \bar{g}_{ij}),$$

hence

$$\begin{aligned} &g^{ij} [-\frac{1}{2} \bar{g}^{\alpha\gamma} \partial_\gamma \partial_\beta \bar{g}_{ij} v^\beta + \bar{g}^{\alpha\gamma} \partial_j \partial_\beta \bar{g}_{i\gamma} v^\beta] \\ &= g^{ij} v^\beta (\partial_\beta [\bar{g}^{\alpha\gamma} (-\frac{1}{2} \partial_\gamma \bar{g}_{ij} + \partial_j \bar{g}_{i\gamma})] - \partial_\beta \bar{g}^{\alpha\gamma} (-\frac{1}{2} \partial_\gamma \bar{g}_{ij} + \partial_j \bar{g}_{i\gamma})) \\ &= g^{ij} v^\beta (\partial_\beta [\bar{\Gamma}_{ij}^\alpha - \frac{1}{2} \bar{g}^{\alpha k} (\partial_i \bar{g}_{jk} + \partial_j \bar{g}_{ik} - \partial_k \bar{g}_{ij})] - \partial_\beta \bar{g}^{\alpha\gamma} (-\frac{1}{2} \partial_\gamma \bar{g}_{ij} + \partial_j \bar{g}_{i\gamma})). \end{aligned}$$

Note that on Σ^n

$$\bar{g}^{\alpha k} = 0,$$

hence we have

$$\begin{aligned} & g^{ij} \left[-\frac{1}{2} \bar{g}^{\alpha\gamma} \partial_\gamma \partial_\beta \bar{g}_{ij} v^\beta + \bar{g}^{\alpha\gamma} \partial_j \partial_\beta \bar{g}_{i\gamma} v^\beta \right] \\ &= g^{ij} v^\beta \partial_\beta \bar{\Gamma}_{ij}^\alpha - \frac{1}{2} g^{ij} v^\beta \partial_\beta \bar{g}^{\alpha k} (\partial_i \bar{g}_{jk} + \partial_j \bar{g}_{ik} - \partial_k \bar{g}_{ij}) \\ &\quad - g^{ij} v^\beta \partial_\beta \bar{g}^{\alpha\gamma} \left(-\frac{1}{2} \partial_\gamma \bar{g}_{ij} + \partial_j \bar{g}_{i\gamma} \right) \\ &= g^{ij} v^\beta \partial_\beta \bar{\Gamma}_{ij}^\alpha + \frac{1}{2} g^{ij} v^\beta \bar{g}^{\alpha\xi} g^{kl} \partial_\beta \bar{g}_{l\xi} (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) \\ &\quad + g^{ij} v^\beta \bar{g}^{\alpha\xi} \bar{g}^{\eta\gamma} \partial_\beta \bar{g}_{\xi\eta} \left(-\frac{1}{2} \partial_\gamma \bar{g}_{ij} + \partial_j \bar{g}_{i\gamma} \right) \\ &= g^{ij} v^\beta \partial_\beta \bar{\Gamma}_{ij}^\alpha + g^{ij} v^\beta \bar{g}^{\alpha\xi} \Gamma_{ij}^l \partial_\beta \bar{g}_{l\xi} + g^{ij} v^\beta \bar{g}^{\alpha\xi} \bar{\Gamma}_{ij}^\eta \partial_\beta \bar{g}_{\xi\eta}. \end{aligned}$$

It then follows from (2.5) and (2.12) that

$$\begin{aligned} & g^{ij} \left[-\frac{1}{2} \bar{g}^{\alpha\gamma} \partial_\gamma \partial_\beta \bar{g}_{ij} v^\beta + \bar{g}^{\alpha\gamma} \partial_j \partial_\beta \bar{g}_{i\gamma} v^\beta \right] \\ &= g^{ij} v^\beta \partial_\beta \bar{\Gamma}_{ij}^\alpha + g^{ij} \Gamma_{ij}^l \bar{g}^{\alpha\xi} v^\beta \partial_\beta \bar{g}_{l\xi} + 2n \bar{g}^{\alpha\xi} v^\beta v^\eta \partial_\beta \bar{g}_{\xi\eta}. \end{aligned}$$

That is (3.6). \square

Lemma 3.4.

$$(3.7) \quad I^\alpha = \Delta^\perp v^\alpha - g^{ij} g^{kl} \langle A_{ik}, v \rangle A_{jl}^\alpha + g^{ij} g^{\alpha\gamma} v^\beta \bar{R}_{i\beta j\gamma} + (4 - 2n) v^\beta v^\eta \bar{\Gamma}_{\beta\eta}^\alpha.$$

Proof. It follows from (3.2), (3.5) and (3.6) that

$$\begin{aligned} (3.8) \quad I^\alpha &= \Delta^\perp v^\alpha - g^{ij} g^{kl} \langle A_{ik}, v \rangle A_{jl}^\alpha \\ &\quad - g^{ij} v^\beta [\partial_j \bar{\Gamma}_{i\beta}^\alpha - \partial_\beta \bar{\Gamma}_{ij}^\alpha + \bar{\Gamma}_{i\beta}^\gamma \bar{\Gamma}_{j\gamma}^\alpha + \bar{\Gamma}_{i\beta}^k \bar{\Gamma}_{jk}^\alpha - \Gamma_{ij}^k \bar{\Gamma}_{k\beta}^\alpha] \\ &\quad + 4 \bar{g}^{\alpha\gamma} \partial_\beta \bar{g}_{\eta\gamma} v^\beta v^\eta - 2 \bar{g}^{\alpha\gamma} \partial_\gamma \bar{g}_{\beta\eta} v^\beta v^\eta. \end{aligned}$$

Note that

$$\begin{aligned} (3.9) \quad \bar{g}^{\alpha\eta} \bar{R}_{j\beta\eta i} &= \bar{g}^{\alpha\eta} \langle \bar{\nabla}_j \bar{\nabla}_\beta \partial_i - \bar{\nabla}_\beta \bar{\nabla}_j \partial_i, \partial_\eta \rangle \\ &= \partial_j \bar{\Gamma}_{\beta i}^\alpha + \bar{\Gamma}_{\beta i}^\gamma \bar{\Gamma}_{j\gamma}^\alpha + \bar{\Gamma}_{\beta i}^k \bar{\Gamma}_{jk}^\alpha - \partial_\beta \bar{\Gamma}_{ij}^\alpha - \bar{\Gamma}_{ij}^k \bar{\Gamma}_{\beta k}^\alpha - \bar{\Gamma}_{ij}^\gamma \bar{\Gamma}_{\beta\gamma}^\alpha. \end{aligned}$$

(3.7) then follows from (3.8) and (3.9). \square

Proposition 3.5. *Let Y^{n+1} , $n \geq 3$, be a minimal surface in (X^{d+1}, g_+) and of the form $Y = (x, u(x, r), r)$ near the boundary of X , where*

$$u(x, r) = \frac{1}{2n} H(x) r^2 + u^{(4)}(x) r^4 + \dots$$

Then we have

$$\begin{aligned} (3.10) \quad 8n(n-2)u^{(4)\alpha} &= \Delta^\perp H^\alpha + g^{ij} g^{kl} \langle A_{ik}, H \rangle A_{jl}^\alpha - \frac{2}{n^2} |H|^2 H^\alpha \\ &\quad + g^{ij} \bar{g}^{\alpha\gamma} H^\beta \bar{R}_{i\beta j\gamma} + 4 \bar{g}^{\alpha\gamma} g_{\beta\gamma}^{(2)} H^\beta + 2 g^{ij} g_{ij}^{(2)} H^\alpha - 2n g^{ik} g^{jl} g_{kl}^{(2)} A_{ij}^\alpha \\ &\quad - n \bar{g}^{\alpha\gamma} g^{ij} (\bar{\nabla}_\gamma g_{ij}^{(2)} - 2 \bar{\nabla}_i g_{j\gamma}^{(2)}) - \frac{n-2}{n} H^\beta H^\eta \bar{\Gamma}_{\beta\eta}^\alpha. \end{aligned}$$

Proof. It follows from (3.1), (3.7) and (3.3) that

$$\begin{aligned}
4(n-2)u^{(4)\alpha} &= -2(n-2)\bar{g}^{\alpha\gamma}g_{\beta\gamma}^{(2)}v^\beta - 8|v|^2v^\alpha + 2g^{ij}g_{ij}^{(2)}v^\alpha \\
&\quad - g^{ik}g^{jl}g_{kl}^{(2)}A_{ij}^\alpha + 2g^{ik}g^{jl}\langle A_{kl}, v \rangle A_{ij}^\alpha \\
&\quad + \Delta^\perp v^\alpha - g^{ij}g^{kl}\langle A_{ik}, v \rangle A_{jl}^\alpha + g^{ij}g^{\alpha\gamma}v^\beta \bar{R}_{i\beta j\gamma} + (4-2n)v^\beta v^\eta \bar{\Gamma}_{\beta\eta}^\alpha \\
&\quad - \frac{1}{2}\bar{g}^{\alpha\gamma}g^{ij}(\bar{\nabla}_\gamma g_{ij}^{(2)} - 2\bar{\nabla}_i g_{j\gamma}^{(2)}) + \bar{g}^{\alpha\gamma}H^\beta g_{\beta\gamma}^{(2)}.
\end{aligned}$$

Using (2.12), we get (3.10). \square

Substituting $g^{(2)} = -\bar{P}$ and using (2.2), one gets (1.10). For $n = 2$, similar calculations imply the following

Proposition 3.6. *Let Y^3 be a minimal surface in (X^{d+1}, g_+) and of the form $Y = (x, u(x, r), r)$ near the boundary of X , where*

$$u(x, r) = \frac{1}{4}H(x)r^2 + w_2(x)r^4 \log r + u^{(4)}(x)r^4 + \dots$$

Then we have

$$\begin{aligned}
-16w_2^\alpha(x) &= \Delta^\perp H^\alpha + g^{ij}g^{kl}\langle A_{ik}, H \rangle A_{jl}^\alpha - \frac{1}{2}|H|^2 H^\alpha \\
(3.11) \quad &\quad + g^{ij}\bar{g}^{\alpha\gamma}H^\beta \bar{W}_{i\beta j\gamma} - 2\bar{g}^{\alpha\gamma}\bar{P}_{\beta\gamma}H^\beta - g^{ij}\bar{P}_{ij}H^\alpha + 4g^{ik}g^{jl}\bar{P}_{kl}A_{ij}^\alpha \\
&\quad + 2\bar{g}^{\alpha\gamma}g^{ij}(\bar{\nabla}_\gamma \bar{P}_{ij} - 2\bar{\nabla}_i \bar{P}_{j\gamma}).
\end{aligned}$$

4. PROOF OF PROPOSITION 1.3

In this section we prove Proposition 1.3. Namely, we calculate Graham-Witten's conformal invariant L_4 for closed four dimensional submanifolds. The volume form and the volume expansion of the minimal surface $Y^{n+1} \hookrightarrow (X^{d+1}, g_+)$ are given by [16]

$$(4.1) \quad d\mu_Y = r^{-n-1}\sqrt{\det \bar{h}}dxdr = r^{-n-1}[v^{(0)} + v^{(2)}r^2 + (\text{even powers}) + v^{(n)}r^n + \dots]d\mu_\Sigma dr,$$

and for n even

$$(4.2) \quad Vol_{g_+}(Y \cap \{r > \epsilon\}) = c_0\epsilon^{-n} + (\text{even powers}) + c_{n-2}\epsilon^{-2} + L_n \log \frac{1}{\epsilon} + c_n + o(1),$$

where

$$(4.3) \quad L_n = \int_{\Sigma^n} v^{(n)} d\mu_\Sigma$$

is invariant under conformal transformations of (M^d, \bar{g}) .

Let $n \geq 4$ and

$$u^\alpha(x, r) = v^\alpha(x)r^2 + w^\alpha(x)r^4 + \dots$$

where $v(x) = \frac{1}{2n}H$ and $w = u^{(4)}$ is given by (3.10). We now compute $v^{(4)}$ in (4.1). Recall that

$$\begin{aligned}
\bar{h}_{rr} &= \bar{h}(Y_r, Y_r) = 1 + \tilde{g}_{\alpha\beta}u_r^\alpha u_r^\beta, \\
\bar{h}_{ir} &= \bar{h}(Y_i, Y_r) = \tilde{g}_{i\alpha}u_r^\alpha + \tilde{g}_{\alpha\beta}u_i^\alpha u_r^\beta,
\end{aligned}$$

and

$$\bar{h}_{ij} = \bar{h}(Y_i, Y_j) = \tilde{g}_{ij} + \tilde{g}_{i\alpha}u_j^\alpha + \tilde{g}_{j\alpha}u_i^\alpha + \tilde{g}_{\alpha\beta}u_i^\alpha u_j^\beta.$$

Hence

$$\begin{aligned}
\bar{h}_{rr} &= 1 + u_r^\alpha u_r^\beta \tilde{g}_{\alpha\beta} \\
&= 1 + (2rv^\alpha + 4r^3w^\alpha)(2rv^\beta + 4r^3w^\beta)(\bar{g}_{\alpha\beta} + g_{\alpha\beta}^{(2)}r^2 + \partial_\gamma \bar{g}_{\alpha\beta} v^\gamma r^2) + o(r^4) \\
&= 1 + 4r^2|v|^2 + 4r^4(v^\alpha v^\beta g_{\alpha\beta}^{(2)} + v^\alpha v^\beta v^\gamma \partial_\gamma \bar{g}_{\alpha\beta} + 4v^\alpha w^\beta \bar{g}_{\alpha\beta}) + o(r^4), \\
\bar{h}_{ir} &= \tilde{g}_{i\alpha} u_r^\alpha + u_i^\alpha u_r^\beta \tilde{g}_{\alpha\beta} \\
&= 2rv^\alpha (g_{i\alpha}^{(2)} r^2 + \partial_\gamma \bar{g}_{i\alpha} v^\gamma r^2) + v_i^\alpha r^2 2v^\beta r \bar{g}_{\alpha\beta} + o(r^3) \\
&= 2r^3 v^\alpha (g_{i\alpha}^{(2)} + \partial_\gamma \bar{g}_{i\alpha} v^\gamma + v_i^\beta \bar{g}_{\alpha\beta}) + o(r^3),
\end{aligned}$$

and

$$\begin{aligned}
\bar{h}_{ij} &= \tilde{g}_{ij} + \tilde{g}_{i\alpha} u_j^\alpha + \tilde{g}_{j\alpha} u_i^\alpha + \tilde{g}_{\alpha\beta} u_i^\alpha u_j^\beta \\
&= g_{ij} + (g_{ij}^{(2)} + \partial_\gamma \bar{g}_{ij} v^\gamma) r^2 + r^4 (g_{ij}^{(4)} + \partial_\beta g_{ij}^{(2)} v^\beta + \partial_\gamma \bar{g}_{ij} w^\gamma + \frac{1}{2} \partial_\beta \partial_\gamma \bar{g}_{ij} v^\beta v^\gamma) \\
&\quad + r^4 (g_{i\alpha}^{(2)} + \partial_\gamma \bar{g}_{i\alpha} v^\gamma) v_j^\alpha + r^4 (g_{j\alpha}^{(2)} + \partial_\gamma \bar{g}_{j\alpha} v^\gamma) v_i^\alpha + r^4 \bar{g}_{\alpha\beta} v_i^\alpha v_j^\beta + o(r^4).
\end{aligned}$$

We rewrite the above as

$$\begin{aligned}
\bar{h}_{rr} &= 1 + ar^2 + br^4, \\
\bar{h}_{ir} &= a_i r^3, \\
\bar{h}_{ij} &= g_{ij} + a_{ij} r^2 + b_{ij} r^4,
\end{aligned}$$

where

$$\begin{aligned}
a &= 4|v|^2, \\
b &= 4(v^\alpha v^\beta g_{\alpha\beta}^{(2)} + v^\alpha v^\beta v^\gamma \partial_\gamma \bar{g}_{\alpha\beta} + 4\langle v, w \rangle), \\
a_i &= 2v^\alpha (g_{i\alpha}^{(2)} + \partial_\gamma \bar{g}_{i\alpha} v^\gamma + v_i^\beta \bar{g}_{\alpha\beta}), \\
a_{ij} &= g_{ij}^{(2)} - 2\langle A_{ij}, v \rangle, \\
b_{ij} &= g_{ij}^{(4)} - 2\langle A_{ij}, w \rangle + \partial_\beta g_{ij}^{(2)} v^\beta + v_i^\alpha g_{j\alpha}^{(2)} + v_j^\alpha g_{i\alpha}^{(2)} \\
&\quad + \bar{g}_{\alpha\beta} v_i^\alpha v_j^\beta + \frac{1}{2} \partial_\beta \partial_\gamma \bar{g}_{ij} v^\beta v^\gamma + \partial_\gamma \bar{g}_{i\alpha} v^\gamma v_j^\alpha + \partial_\gamma \bar{g}_{j\alpha} v^\gamma v_i^\alpha.
\end{aligned}$$

Let $A = (a_{ij})$, $B = (b_{ij})$. Then

$$\begin{aligned}
\det \bar{h} &= (1 + ar^2 + br^4) \det(\bar{h}_{ij}) + o(r^4) \\
&= \det(\bar{h}_{ij}) + ar^2 \det(g_{ij} + a_{ij} r^2) + br^4 \det(g_{ij}) + o(r^4) \\
&= \det(\bar{h}_{ij}) + ar^2 \det(g_{ij})(1 + g^{ij} a_{ij} r^2) + br^4 \det(g_{ij}) + o(r^4).
\end{aligned}$$

Note that

$$\det(\bar{h}_{ij}) = \det(g_{ij})[1 + g^{ij} a_{ij} r^2 + (g^{ij} b_{ij} + \sigma_2(g^{-1}A))r^4] + o(r^4).$$

Therefore,

$$\det \bar{h} = \det(g_{ij})(1 + Cr^2 + Dr^4) + o(r^4),$$

where

$$C = a + g^{ij}a_{ij},$$

$$D = \text{atr}(g^{-1}A) + \sigma_2(g^{-1}A) + b + \text{tr}(g^{-1}B).$$

Then

$$(4.4) \quad \begin{aligned} d\mu_Y &= r^{-(n+1)} \sqrt{\det \bar{h}} dx dr \\ &= r^{-(n+1)} \left(1 + \frac{C}{2}r^2 + \frac{1}{2}\left(D - \frac{C^2}{4}\right)r^4 + o(r^4)\right) d\mu_\Sigma dr, \end{aligned}$$

and the coefficient $v^{(4)}$ in (4.1) is

$$(4.5) \quad v^{(4)} = \frac{1}{2}\left(D - \frac{C^2}{4}\right).$$

Proposition 4.1. *Let Y^{n+1} , $n \geq 4$, be a minimal surface in (X^{d+1}, g_+) . Then we have*

$$(4.6) \quad \begin{aligned} 2v^{(4)} &= 4|v|^2(g^{ij}g_{ij}^{(2)} - 4n|v|^2) + \frac{1}{2}(g^{ij}g_{ij}^{(2)} - 4n|v|^2)^2 - \frac{1}{2}|g_{ij}^{(2)} - 2\langle A_{ij}, v \rangle|_g^2 \\ &\quad - \frac{1}{4}(g^{ij}g_{ij}^{(2)} - 4(n-1)|v|^2)^2 + 4g_{\alpha\beta}^{(2)}v^\alpha v^\beta - 4(n-4)\langle v, w \rangle + g^{ij}g_{ij}^{(4)} \\ &\quad + I + II, \end{aligned}$$

where

$$(4.7) \quad I := g^{ij}\bar{g}_{\alpha\beta}v_i^\alpha v_j^\beta + \frac{1}{2}g^{ij}\partial_\beta\partial_\gamma\bar{g}_{ij}v^\beta v^\gamma + 2g^{ij}\partial_\gamma\bar{g}_{i\alpha}v^\gamma v_j^\alpha + 4v^\alpha v^\beta v^\gamma\partial_\gamma\bar{g}_{\alpha\beta},$$

and

$$(4.8) \quad II := g^{ij}(\partial_\beta g_{ij}^{(2)}v^\beta + 2g_{i\alpha}^{(2)}v_j^\alpha).$$

We first deal with (4.8).

Lemma 4.2. *We have*

$$(4.9) \quad II = g^{ij}v^\beta(\bar{\nabla}_\beta g_{ij}^{(2)} - 2\bar{\nabla}_j g_{i\beta}^{(2)}) - 4ng_{\alpha\beta}^{(2)}v^\alpha v^\beta + 2g^{ij}\nabla_j(v^\beta g_{i\beta}^{(2)}).$$

Proof. Note that

$$\begin{aligned} g^{ij}\nabla_j(v^\beta g_{i\beta}^{(2)}) &= g^{ij}[\partial_j(v^\beta g_{i\beta}^{(2)}) - \Gamma_{ij}^k v^\beta g_{k\beta}^{(2)}] \\ &= g^{ij}v_j^\beta g_{i\beta}^{(2)} + g^{ij}v^\beta \partial_j g_{i\beta}^{(2)} - g^{ij}\Gamma_{ij}^k v^\beta g_{k\beta}^{(2)}, \end{aligned}$$

hence

$$\begin{aligned} &g^{ij}(\partial_\beta g_{ij}^{(2)}v^\beta + 2g_{i\beta}^{(2)}v_j^\beta) - 2g^{ij}\nabla_j(v^\beta g_{i\beta}^{(2)}) \\ &= g^{ij}v^\beta(\partial_\beta g_{ij}^{(2)} - 2\partial_j g_{i\beta}^{(2)} + 2\Gamma_{ij}^k g_{k\beta}^{(2)}) \\ &= g^{ij}v^\beta(\partial_\beta g_{ij}^{(2)} - \partial_j g_{i\beta}^{(2)} - \partial_i g_{j\beta}^{(2)} + 2\Gamma_{ij}^k g_{k\beta}^{(2)}). \end{aligned}$$

As we have shown in (3.4), we have

$$\partial_\beta g_{ij}^{(2)} - \partial_j g_{i\beta}^{(2)} - \partial_i g_{j\beta}^{(2)} = \bar{\nabla}_\beta g_{ij}^{(2)} - \bar{\nabla}_j g_{i\beta}^{(2)} - \bar{\nabla}_i g_{j\beta}^{(2)} - 2\bar{\Gamma}_{ij}^D g_{D\beta}^{(2)}.$$

Therefore,

$$\begin{aligned}
& g^{ij}(\partial_\beta g_{ij}^{(2)} v^\beta + 2g_{i\beta}^{(2)} v_j^\beta) - 2g^{ij} \nabla_j (v^\beta g_{i\beta}^{(2)}) \\
&= g^{ij} v^\beta (\bar{\nabla}_\beta g_{ij}^{(2)} - 2\bar{\nabla}_j g_{i\beta}^{(2)} - 2\bar{\Gamma}_{ij}^D g_{D\beta}^{(2)} + 2\Gamma_{ij}^k g_{k\beta}^{(2)}) \\
&= g^{ij} v^\beta (\bar{\nabla}_\beta g_{ij}^{(2)} - 2\bar{\nabla}_j g_{i\beta}^{(2)} - 2\bar{\Gamma}_{ij}^\alpha g_{\alpha\beta}^{(2)}) \\
&= g^{ij} v^\beta (\bar{\nabla}_\beta g_{ij}^{(2)} - 2\bar{\nabla}_j g_{i\beta}^{(2)}) - 2H^\alpha v^\beta g_{\alpha\beta}^{(2)}.
\end{aligned}$$

□

Lemma 4.3. *We have*

$$\begin{aligned}
(4.10) \quad I &= -\langle \Delta^\perp v, v \rangle + g^{ij} g^{kl} \langle A_{ik}, v \rangle \langle A_{jl}, v \rangle - g^{ij} v^\alpha v^\beta \bar{R}_{i\alpha j\beta} \\
&\quad - 2(n-4) \bar{g}_{\alpha\xi} v^\xi v^\beta v^\gamma \bar{\Gamma}_{\beta\gamma}^\alpha + g^{ij} \nabla_j (\bar{g}_{\alpha\beta} v_i^\alpha v^\beta + \partial_\alpha \bar{g}_{i\beta} v^\alpha v^\beta).
\end{aligned}$$

Proof. Note that

$$g^{ij} \bar{g}_{\alpha\beta} v_i^\alpha v_j^\beta = g^{ij} \nabla_j (\bar{g}_{\alpha\beta} v_i^\alpha v^\beta) + g^{ij} \Gamma_{ij}^k \bar{g}_{\alpha\beta} v_k^\alpha v^\beta - g^{ij} \partial_j \bar{g}_{\alpha\beta} v_i^\alpha v^\beta - g^{ij} \bar{g}_{\alpha\beta} \partial_j v_i^\alpha v^\beta.$$

It then follows from (3.5):

$$\begin{aligned}
\partial_j v_i^\alpha &= \nabla_j^\perp \nabla_i^\perp v^\alpha - g^{kl} \langle A_{ik}, v \rangle A_{jl}^\alpha - v_i^\beta \bar{\Gamma}_{j\beta}^\alpha - v_j^\beta \bar{\Gamma}_{i\beta}^\alpha + v_k^\alpha \Gamma_{ij}^k \\
&\quad - v^\beta (\partial_j \bar{\Gamma}_{i\beta}^\alpha + \bar{\Gamma}_{i\beta}^\gamma \bar{\Gamma}_{j\gamma}^\alpha + \bar{\Gamma}_{i\beta}^k \bar{\Gamma}_{jk}^\alpha - \Gamma_{ij}^k \bar{\Gamma}_{k\beta}^\alpha)
\end{aligned}$$

and (4.7):

$$I = g^{ij} \bar{g}_{\alpha\beta} v_i^\alpha v_j^\beta + \frac{1}{2} g^{ij} \partial_\beta \partial_\gamma \bar{g}_{ij} v^\beta v^\gamma + 2g^{ij} \partial_\gamma \bar{g}_{i\alpha} v^\alpha v_j^\gamma + 4v^\alpha v^\beta v^\gamma \partial_\gamma \bar{g}_{\alpha\beta},$$

that

$$\begin{aligned}
I &= -\langle \Delta^\perp v, v \rangle + g^{ij} g^{kl} \langle A_{ik}, v \rangle \langle A_{jl}, v \rangle + g^{ij} \nabla_j (\bar{g}_{\alpha\beta} v_i^\alpha v^\beta) \\
&\quad + \frac{1}{2} g^{ij} \partial_\beta \partial_\gamma \bar{g}_{ij} v^\beta v^\gamma + 2g^{ij} \partial_\gamma \bar{g}_{i\alpha} v^\alpha v_j^\gamma + 4v^\alpha v^\beta v^\gamma \partial_\gamma \bar{g}_{\alpha\beta} - g^{ij} \partial_j \bar{g}_{\alpha\beta} v_i^\alpha v^\beta \\
&\quad + g^{ij} \bar{g}_{\alpha\xi} v^\xi [2v_i^\beta \bar{\Gamma}_{j\beta}^\alpha + v^\beta (\partial_j \bar{\Gamma}_{i\beta}^\alpha + \bar{\Gamma}_{i\beta}^\gamma \bar{\Gamma}_{j\gamma}^\alpha + \bar{\Gamma}_{i\beta}^k \bar{\Gamma}_{jk}^\alpha - \Gamma_{ij}^k \bar{\Gamma}_{k\beta}^\alpha)]
\end{aligned}$$

Note that

$$\begin{aligned}
& 2g^{ij} \partial_\gamma \bar{g}_{i\alpha} v^\alpha v_j^\gamma - g^{ij} \partial_j \bar{g}_{\alpha\beta} v_i^\alpha v^\beta + 2g^{ij} \bar{g}_{\alpha\xi} v^\xi v_i^\beta \bar{\Gamma}_{j\beta}^\alpha \\
&= g^{ij} v_i^\alpha v^\gamma [2\partial_\gamma \bar{g}_{j\alpha} - \partial_j \bar{g}_{\alpha\gamma} + 2\bar{g}_{\beta\gamma} \bar{\Gamma}_{j\alpha}^\beta] \\
&= g^{ij} v_i^\alpha v^\gamma [\partial_\gamma \bar{g}_{j\alpha} + \partial_\alpha \bar{g}_{j\gamma}] \\
&= g^{ij} \partial_\gamma \bar{g}_{j\alpha} \partial_i (v^\alpha v^\gamma) \\
&= g^{ij} \nabla_i (\partial_\gamma \bar{g}_{j\alpha} v^\alpha v^\gamma) + g^{ij} \Gamma_{ij}^k \partial_\gamma \bar{g}_{k\alpha} v^\alpha v^\gamma - g^{ij} \partial_i \partial_\gamma \bar{g}_{j\alpha} v^\alpha v^\gamma.
\end{aligned}$$

Hence

$$\begin{aligned}
I &= -\langle \Delta^\perp v, v \rangle + g^{ij} g^{kl} \langle A_{ik}, v \rangle \langle A_{jl}, v \rangle + g^{ij} \nabla_j (\bar{g}_{\alpha\beta} v_i^\alpha v^\beta + \partial_\gamma \bar{g}_{i\alpha} v^\alpha v^\gamma) \\
&\quad + \frac{1}{2} g^{ij} \partial_\beta \partial_\gamma \bar{g}_{ij} v^\beta v^\gamma - g^{ij} \partial_j \partial_\gamma \bar{g}_{i\beta} v^\beta v^\gamma + g^{ij} \Gamma_{ij}^k \partial_\gamma \bar{g}_{k\alpha} v^\alpha v^\gamma + 4v^\alpha v^\beta v^\gamma \partial_\gamma \bar{g}_{\alpha\beta} \\
&\quad + g^{ij} \bar{g}_{\alpha\xi} v^\xi v^\beta (\partial_j \bar{\Gamma}_{i\beta}^\alpha + \bar{\Gamma}_{i\beta}^\gamma \bar{\Gamma}_{j\gamma}^\alpha + \bar{\Gamma}_{i\beta}^k \bar{\Gamma}_{jk}^\alpha - \Gamma_{ij}^k \bar{\Gamma}_{k\beta}^\alpha).
\end{aligned}$$

Recall (3.6):

$$\begin{aligned} & g^{ij} \left[-\frac{1}{2} \bar{g}^{\alpha\gamma} \partial_\gamma \partial_\beta \bar{g}_{ij} v^\beta + \bar{g}^{\alpha\gamma} \partial_j \partial_\beta \bar{g}_{i\gamma} v^\beta \right] \\ &= g^{ij} v^\beta \partial_\beta \bar{\Gamma}_{ij}^\alpha + g^{ij} \Gamma_{ij}^l \bar{g}^{\alpha\xi} v^\beta \partial_\beta \bar{g}_{l\xi} + 2n \bar{g}^{\alpha\xi} v^\beta v^\eta \partial_\beta \bar{g}_{\xi\eta}, \end{aligned}$$

hence

$$\begin{aligned} & \frac{1}{2} g^{ij} \partial_\beta \partial_\gamma \bar{g}_{ij} v^\beta v^\gamma - g^{ij} \partial_j \partial_\gamma \bar{g}_{i\beta} v^\beta v^\gamma \\ &= -\bar{g}_{\alpha\xi} v^\xi g^{ij} \left[-\frac{1}{2} \bar{g}^{\alpha\gamma} \partial_\gamma \partial_\beta \bar{g}_{ij} v^\beta + \bar{g}^{\alpha\gamma} \partial_j \partial_\beta \bar{g}_{i\gamma} v^\beta \right] \\ &= -g^{ij} \bar{g}_{\alpha\xi} v^\xi v^\beta \partial_\beta \bar{\Gamma}_{ij}^\alpha - g^{ij} \Gamma_{ij}^l v^\xi v^\beta \partial_\beta \bar{g}_{l\xi} - 2n v^\xi v^\beta v^\eta \partial_\beta \bar{g}_{\xi\eta}. \end{aligned}$$

Then

$$\begin{aligned} I &= -\langle \Delta^\perp v, v \rangle + g^{ij} g^{kl} \langle A_{ik}, v \rangle \langle A_{jl}, v \rangle + g^{ij} \nabla_j (\bar{g}_{\alpha\beta} v_i^\alpha v^\beta + \partial_\gamma \bar{g}_{i\alpha} v^\alpha v^\gamma) \\ &\quad + g^{ij} \bar{g}_{\alpha\xi} v^\xi v^\beta (\partial_j \bar{\Gamma}_{i\beta}^\alpha + \bar{\Gamma}_{i\beta}^\gamma \bar{\Gamma}_{j\gamma}^\alpha + \bar{\Gamma}_{i\beta}^k \bar{\Gamma}_{jk}^\alpha - \partial_\beta \bar{\Gamma}_{ij}^\alpha - \Gamma_{ij}^k \bar{\Gamma}_{k\beta}^\alpha) \\ &\quad + (4 - 2n) v^\alpha v^\beta v^\gamma \partial_\gamma \bar{g}_{\alpha\beta}. \end{aligned}$$

Recall (3.9):

$$\bar{g}^{\alpha\eta} \bar{R}_{j\beta\eta i} = \partial_j \bar{\Gamma}_{\beta i}^\alpha + \bar{\Gamma}_{\beta i}^\gamma \bar{\Gamma}_{j\gamma}^\alpha + \bar{\Gamma}_{\beta i}^k \bar{\Gamma}_{jk}^\alpha - \partial_\beta \bar{\Gamma}_{ij}^\alpha - \bar{\Gamma}_{ij}^k \bar{\Gamma}_{\beta k}^\alpha - \bar{\Gamma}_{ij}^\gamma \bar{\Gamma}_{\beta\gamma}^\alpha,$$

and $\bar{\Gamma}_{ij}^\gamma = A_{ij}^\gamma$, $H^\gamma = 2n v^\gamma$, so we have

$$\begin{aligned} I &= -\langle \Delta^\perp v, v \rangle + g^{ij} g^{kl} \langle A_{ik}, v \rangle \langle A_{jl}, v \rangle + g^{ij} \nabla_j (\bar{g}_{\alpha\beta} v_i^\alpha v^\beta + \partial_\gamma \bar{g}_{i\alpha} v^\alpha v^\gamma) \\ &\quad - g^{ij} v^\alpha v^\beta \bar{R}_{i\alpha j\beta} - 2(n-4) \bar{g}_{\alpha\xi} v^\xi v^\beta v^\gamma \bar{\Gamma}_{\beta\gamma}^\alpha. \end{aligned}$$

□

It follows from (4.6), (4.10) and (4.9) the following

Proposition 4.4. *Let Y^{n+1} , $n \geq 4$, be a minimal surface in (X^{d+1}, g_+) . The coefficient $v^{(4)}$ in (4.1) is given by*

$$\begin{aligned} 2v^{(4)} &= -\langle \Delta^\perp v, v \rangle - g^{ik} g^{jl} \langle A_{ij}, v \rangle \langle A_{kl}, v \rangle + 4(n^2 - 2n - 1) |v|^4 \\ &\quad + \frac{1}{4} (g^{ij} g_{ij}^{(2)})^2 - \frac{1}{2} |g_{ij}^{(2)}|^2_g + g^{ij} g_{ij}^{(4)} \\ &\quad + 2g^{ik} g^{jl} g_{ij}^{(2)} \langle A_{kl}, v \rangle - 2(n-1) g^{ij} g_{ij}^{(2)} |v|^2 - 4(n-1) g_{\alpha\beta}^{(2)} v^\alpha v^\beta - g^{ij} \bar{R}_{i\alpha j\beta} v^\alpha v^\beta \\ (4.11) \quad &\quad + g^{ij} v^\beta (\bar{\nabla}_\beta g_{ij}^{(2)} - 2\bar{\nabla}_i g_{j\beta}^{(2)}) \\ &\quad - 4(n-4) \langle v, w \rangle - 2(n-4) \bar{g}_{\alpha\xi} v^\xi v^\beta v^\gamma \bar{\Gamma}_{\beta\gamma}^\alpha \\ &\quad + g^{ij} \nabla_j (\bar{g}_{\alpha\beta} v_i^\alpha v^\beta + \partial_\alpha \bar{g}_{i\beta} v^\alpha v^\beta + 2g_{i\alpha}^{(2)} v^\alpha), \end{aligned}$$

where $v = \frac{1}{2n} H$ and $w = u^{(4)}$ is given by (3.10).

Notice that the second last line in (4.11) vanishes for $n = 4$ and for $n \geq 5$ the part involving $\bar{\Gamma}$ cancels. Note also that the last line is a divergence which is independent of the choice of local coordinates (x^i, y^α) . By integrating (4.11) over a closed submanifold Σ^4 and using (2.1), (2.2) and (2.3), we obtain the expression (1.18) of Graham-Witten's conformal invariant L_4 .

5. L_4 FOR CLOSED SUBMANIFOLDS OF EUCLIDEAN SPACES

In this section, we assume $n = 4, d \geq 5$ and the ambient space is $(M^d, \bar{g}) = \mathbb{R}^d$. It follows from (1.18) that

$$128L_4 = \int_{\Sigma} (|\nabla^\perp H|^2 - g^{ik}g^{jl}\langle A_{ij}, H \rangle \langle A_{kl}, H \rangle + \frac{7}{16}|H|^4) d\mu_g.$$

We now consider the following functional for closed four dimensional submanifolds $F : \Sigma \rightarrow \mathbb{R}^d$

$$\mathcal{L}_4(\Sigma^4) = \frac{1}{2} \int_{\Sigma} (|\nabla^\perp H|^2 - g^{ik}g^{jl}\langle A_{ij}, H \rangle \langle A_{kl}, H \rangle + \frac{7}{16}|H|^4) d\mu_g.$$

In the sequel we use local coordinates (x^i) of Σ which is normal at the point of consideration with respect to the induced metric

$$g_{ij} := \langle F_i, F_j \rangle.$$

Proposition 5.1. $F : \Sigma^4 \hookrightarrow \mathbb{R}^d$ is a critical point of \mathcal{L}_4 if and only if

$$(5.1) \quad \mathcal{E} = 0,$$

where

$$(5.2) \quad \begin{aligned} \mathcal{E} = & \Delta^\perp(\Delta^\perp H + \langle A_{ij}, H \rangle A_{ij} - \frac{7}{8}|H|^2 H) \\ & + \langle \Delta^\perp H, A_{ij} \rangle A_{ij} + \langle \Delta^\perp H, H \rangle H - \langle \nabla_i^\perp H, \nabla_j^\perp H \rangle A_{ij} + \frac{3}{2}|\nabla^\perp H|^2 H \\ & + 2\langle A_{ij}, H \rangle \Delta^\perp A_{ij} + 2\langle A_{ij}, \nabla_i^\perp H \rangle \nabla_j^\perp H + 3\langle H, \nabla_i^\perp H \rangle \nabla_i^\perp H \\ & - \langle A_{ij}, H \rangle \langle A_{jk}, H \rangle A_{ik} - \frac{1}{2}\langle A_{ij}, H \rangle \langle A_{ij}, H \rangle H + 3\langle A_{ij}, H \rangle \langle A_{ij}, A_{kl} \rangle A_{kl} \\ & - \frac{7}{8}|H|^2 \langle A_{ij}, H \rangle A_{ij} + \frac{7}{32}|H|^4 H \\ & - 2\langle A_{ij}, H \rangle \langle A_{ik}, A_{jl} \rangle A_{kl} + 2\langle A_{ij}, H \rangle \langle A_{ik}, A_{kl} \rangle A_{jl}. \end{aligned}$$

Proof. Let $X \in T^\perp \Sigma$ be a variation field. One has

$$\delta g_{ij} = -2\langle A_{ij}, X \rangle,$$

$$\delta A_{ij} = \nabla_i^\perp \nabla_j^\perp X - \langle A_{jk}, X \rangle A_{ik} - \langle A_{ij}, \nabla_k^\perp X \rangle F_k,$$

$$\delta H = \Delta^\perp X + \langle A_{kl}, X \rangle A_{kl} - \langle H, \nabla_k^\perp X \rangle F_k,$$

and

$$[\delta \nabla_i^\perp H]^\perp = \nabla_i^\perp (\Delta^\perp X + \langle A_{kl}, X \rangle A_{kl}) - \langle H, \nabla_k^\perp X \rangle A_{ik} + \langle A_{ik}, H \rangle \nabla_k^\perp X.$$

Hence

$$\begin{aligned}
& \delta \int_{\Sigma} |\nabla^{\perp} H|_g^2 d\mu_g \\
&= \int_{\Sigma} (2\langle A_{ij}, X \rangle \langle \nabla_i^{\perp} H, \nabla_j^{\perp} H \rangle - |\nabla^{\perp} H|^2 \langle H, X \rangle) d\mu \\
&\quad + \int_{\Sigma} 2\langle \nabla_i^{\perp} (\Delta^{\perp} X + \langle A_{kl}, X \rangle A_{kl}) - \langle H, \nabla_k^{\perp} X \rangle A_{ik} + \langle A_{ik}, H \rangle \nabla_k^{\perp} X, \nabla_i^{\perp} H \rangle d\mu \\
&= \int_{\Sigma} \langle -2(\Delta^{\perp})^2 H - 2\langle \Delta^{\perp} H, A_{ij} \rangle A_{ij} + 2\langle \nabla_i^{\perp} H, \nabla_j^{\perp} H \rangle A_{ij} - |\nabla^{\perp} H|^2 H, X \rangle d\mu \\
&\quad + \int_{\Sigma} 2\langle \nabla_k^{\perp} (\langle A_{ik}, \nabla_i^{\perp} H \rangle H - \langle A_{ik}, H \rangle \nabla_i^{\perp} H), X \rangle d\mu.
\end{aligned}$$

We then compute

$$\begin{aligned}
& \delta \int_{\Sigma} (-g^{ik} g^{jl} \langle A_{ij}, H \rangle \langle A_{kl}, H \rangle) d\mu_g \\
&= \int_{\Sigma} \langle -4\langle A_{ij}, H \rangle \langle A_{jk}, H \rangle A_{ik} + \langle A_{ij}, H \rangle \langle A_{ij}, H \rangle H, X \rangle d\mu \\
&\quad - 2 \int_{\Sigma} \langle A_{ij}, H \rangle \langle \nabla_i^{\perp} \nabla_j^{\perp} X - \langle A_{jk}, X \rangle A_{ik}, H \rangle d\mu \\
&\quad - 2 \int_{\Sigma} \langle A_{ij}, H \rangle \langle A_{ij}, \Delta^{\perp} X + \langle A_{kl}, X \rangle A_{kl} \rangle d\mu \\
&= \int_{\Sigma} \langle -2\nabla_j^{\perp} \nabla_i^{\perp} (\langle A_{ij}, H \rangle H) - 2\Delta^{\perp} (\langle A_{ij}, H \rangle A_{ij}), X \rangle d\mu \\
&\quad + \int_{\Sigma} \langle -2\langle A_{ij}, H \rangle \langle A_{jk}, H \rangle A_{ik} + \langle A_{ij}, H \rangle \langle A_{ij}, H \rangle H - 2\langle A_{ij}, H \rangle \langle A_{ij}, A_{kl} \rangle A_{kl}, X \rangle d\mu,
\end{aligned}$$

and

$$\begin{aligned}
\delta \int_{\Sigma} \frac{7}{16} |H|^4 d\mu_g &= \int_{\Sigma} \frac{7}{4} |H|^2 \langle H, \Delta^{\perp} X + \langle A_{ij}, X \rangle A_{ij} \rangle d\mu - \int_{\Sigma} \frac{7}{16} |H|^4 \langle H, X \rangle d\mu \\
&= \int_{\Sigma} \langle \frac{7}{4} \Delta^{\perp} (|H|^2 H) + \frac{7}{4} |H|^2 \langle A_{ij}, H \rangle A_{ij} - \frac{7}{16} |H|^4 H, X \rangle d\mu.
\end{aligned}$$

Therefore

$$(5.3) \quad \delta \mathcal{L}_4(X) = - \int_{\Sigma} \langle \mathcal{E}, X \rangle d\mu,$$

where

$$\begin{aligned}
\mathcal{E} &= (\Delta^{\perp})^2 H + \langle \Delta^{\perp} H, A_{ij} \rangle A_{ij} - \langle \nabla_i^{\perp} H, \nabla_j^{\perp} H \rangle A_{ij} + \frac{1}{2} |\nabla^{\perp} H|^2 H \\
&\quad - \nabla_k^{\perp} (\langle A_{ik}, \nabla_i^{\perp} H \rangle H) + \nabla_k^{\perp} (\langle A_{ik}, H \rangle \nabla_i^{\perp} H) \\
&\quad + \nabla_i^{\perp} \nabla_j^{\perp} (\langle A_{ij}, H \rangle H) + \Delta^{\perp} (\langle A_{ij}, H \rangle A_{ij}) - \frac{7}{8} \Delta^{\perp} (|H|^2 H) \\
&\quad + \langle A_{ij}, H \rangle \langle A_{jk}, H \rangle A_{ik} - \frac{1}{2} \langle A_{ij}, H \rangle \langle A_{ij}, H \rangle H + \langle A_{ij}, H \rangle \langle A_{ij}, A_{kl} \rangle A_{kl} \\
&\quad - \frac{7}{8} |H|^2 \langle A_{ij}, H \rangle A_{ij} + \frac{7}{32} |H|^4 H.
\end{aligned}$$

Note that

$$\begin{aligned} & -\nabla_k^\perp(\langle A_{ik}, \nabla_i^\perp H \rangle H) + \nabla_k^\perp(\langle A_{ik}, H \rangle \nabla_i^\perp H) + \nabla_i^\perp \nabla_j^\perp(\langle A_{ij}, H \rangle H) \\ & = 2\langle A_{ij}, H \rangle \nabla_i^\perp \nabla_j^\perp H + 2\langle A_{ij}, \nabla_i^\perp H \rangle \nabla_j^\perp H + 3\langle H, \nabla_i^\perp H \rangle \nabla_i^\perp H \\ & \quad + \langle \Delta^\perp H, H \rangle H + |\nabla^\perp H|^2 H, \end{aligned}$$

and

$$\nabla_i^\perp \nabla_j^\perp H = \Delta^\perp A_{ij} + \langle A_{ij}, A_{kl} \rangle A_{kl} - \langle A_{ik}, A_{jl} \rangle A_{kl} + \langle A_{ik}, A_{kl} \rangle A_{jl} - \langle A_{ik}, H \rangle A_{jk},$$

hence

$$\begin{aligned} \mathcal{E} & = \Delta^\perp(\Delta^\perp H + \langle A_{ij}, H \rangle A_{ij} - \frac{7}{8}|H|^2 H) \\ & \quad + \langle \Delta^\perp H, A_{ij} \rangle A_{ij} + \langle \Delta^\perp H, H \rangle H - \langle \nabla_i^\perp H, \nabla_j^\perp H \rangle A_{ij} + \frac{3}{2}|\nabla^\perp H|^2 H \\ & \quad + 2\langle A_{ij}, H \rangle \Delta^\perp A_{ij} + 2\langle A_{ij}, \nabla_i^\perp H \rangle \nabla_j^\perp H + 3\langle H, \nabla_i^\perp H \rangle \nabla_i^\perp H \\ & \quad - \langle A_{ij}, H \rangle \langle A_{jk}, H \rangle A_{ik} - \frac{1}{2}\langle A_{ij}, H \rangle \langle A_{ij}, H \rangle H + 3\langle A_{ij}, H \rangle \langle A_{ij}, A_{kl} \rangle A_{kl} \\ & \quad - \frac{7}{8}|H|^2 \langle A_{ij}, H \rangle A_{ij} + \frac{7}{32}|H|^4 H \\ & \quad + 2\langle A_{ij}, H \rangle (-\langle A_{ik}, A_{jl} \rangle A_{kl} + \langle A_{ik}, A_{kl} \rangle A_{jl}). \end{aligned}$$

□

In general, \mathcal{L}_4 is unbounded from below and above even for closed four dimensional submanifolds with a fixed topology. We search for critical points of \mathcal{L}_4 of the form

$$(5.4) \quad \mathbb{S}^{k_1}(r_1) \times \cdots \times \mathbb{S}^{k_m}(r_m),$$

where $r_i > 0, i = 1, \dots, m$, is the radius of a round sphere $\mathbb{S}^{k_i} \hookrightarrow \mathbb{R}^{k_i+1}$, and $k_1 \geq k_2 \geq \cdots \geq k_m, k_1 + \cdots + k_m = 4$. Submanifolds of the form (5.4) have parallel second fundamental form and parallel mean curvature in the normal bundle. We fix $r_1 = 1$.

For $m = 1$, we have the critical point of the round sphere and

$$\mathcal{L}_4(\mathbb{S}^4) = 24Vol(\mathbb{S}^4) = 64\pi^2.$$

For $m = 2$ and $k_1 = 3$, we have two solutions of the form (5.4) to (5.1)

$$\mathcal{L}_4[\mathbb{S}^3(1) \times \mathbb{S}^1(\frac{1}{\sqrt{3}})] = 18\sqrt{3}\pi^3, \quad \mathcal{L}_4[\mathbb{S}^3(1) \times \mathbb{S}^1(\sqrt{\frac{3}{5}})] = 8\sqrt{15}\pi^3.$$

For $m = 2$ and $k_1 = 2$, we have one solution of the form (5.4) to (5.1)

$$\mathcal{L}_4[\mathbb{S}^2(1) \times \mathbb{S}^2(1)] = 96\pi^2.$$

For $m = 3$, we have two solutions of the form (5.4) to (5.1)

$$\mathcal{L}_4[\mathbb{S}^2(1) \times \mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^1(\frac{1}{\sqrt{2}})] = 48\pi^3, \quad \mathcal{L}_4[\mathbb{S}^2(1) \times \mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^1(\frac{3}{\sqrt{10}})] = \frac{64\sqrt{5}}{3}\pi^3.$$

For $m = 4$, we have two solutions of the form (5.4) to (5.1)

$$\mathcal{L}_4[\mathbb{S}^1(1) \times \mathbb{S}^1(1) \times \mathbb{S}^1(1) \times \mathbb{S}^1(1)] = 24\pi^4, \quad \mathcal{L}_4[\mathbb{S}^1(1) \times \mathbb{S}^1(1) \times \mathbb{S}^1(1) \times \mathbb{S}^1(\frac{3}{\sqrt{5}})] = \frac{32\sqrt{5}}{3}\pi^4.$$

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